Fachbereich Mathematik
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5. Problems for Manifolds

Problem 17 – Quiz:
Recall the following definitions:
• tangent vector • vector field • differentiable map and differential • immersion and embedding (is an injective immersion an embedding?) • submanifold

Problem 18 – Lie bracket of vector fields:
Prove $[fX, gY] = fg[X,Y] + f(\partial_X g)Y - g(\partial_Y f)X$ for all $f, g \in \mathcal{D}(M)$, $X, Y \in \mathcal{V}(M)$.

Problem 19 – Expansion of a flow:
For $X \in \mathcal{V}(\mathbb{R}^n)$ verify the expansion $\varphi_t(p) = p + tX(p) + O(t^2)$ at $t = 0$.

Problem 20 – Lie subalgebras:

a) An $n \times n$ matrix is skew-Hermitian if ${}^t\overline{A} = -A$. Prove that the set of skew-Hermitian matrices is closed under $[A, B] = AB - BA$.

b) Find another such matrix algebra.
Hint: What is the trace of $[A, B]$?

Problem 21 – Determination of flows:
Determine the flow of the following vector fields on $\mathbb{R}^2$:
a) $X = xe_1 + 2ye_2$, b) $Y = xe_1 - ye_2$, c) $Z = xe_2 + ye_1$

Problem 22 – Flows on compact manifolds:
Recall that a flow φ of a vector field $X \in \mathcal{V}(M)$ is global if $\varphi(t, p)$ exists for all $t \in \mathbb{R}$ and $p \in M$. Prove that φ is global if M is compact and $X \in \mathcal{V}(M)$.
Hint: On a compact manifold, each sequence has a convergent subsequence.

Problem 23 – Index of a vector field on a surface:
Suppose $X \in \mathcal{V}(\mathbb{R}^2)$ has only a discrete set of zeros Z. For any differentiable loop $c(t)$ in $\mathbb{R}^2 \setminus Z$, let $\varphi(t) = \angle(X(c(t)), E(c(t))$ be continuous, and define the number $i(X, c) := \frac{1}{2\pi} \int \varphi'(t)\, dt$ as the total change of angle along c which X makes against a constant vector field $E \in \mathcal{V}(\mathbb{R}^n)$.

a) Prove that $i(X, c)$ does not depend on E. (Do you get the same number for an arbitrary choice of $E \in \mathcal{V}(M)$?)

b) Prove that curves c_1, c_2 which are (differentiably) homotopic in $\mathbb{R}^2 \setminus Z$ have the same index, $i(X, c_1) = i(X, c_2)$.

c) Let $p \in Z$ and c be a curve in $\mathbb{R}^2 \setminus Z$ which is null homotopic in $\{p\} \cup \{\mathbb{R}^2 \setminus Z\}$ and has winding number $+1$ about p. Then the index $j(X, p)$ of X at p is defined by $j(X, p) := i(X, c)$. (Compare with pictures on p.2.)

d) If you attend Riemannian geometry: Note that the angle is only defined in terms of the standard Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$. Prove that if g is any other Riemannian metric on $\mathbb{R}^2$, the similarly defined number $i(X, c) := i(g, X, c)$ agrees.

e) Extensions: Reason that $i(X, c)$ is defined on differentiable manifolds M. Do you have any idea for a similar number in higher dimensions?