Fachbereich Mathematik
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TECHNISCHE
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4. Problems for Manifolds

Problem 12 – Manifolds as metric spaces:

a) Consider an immersion $f\colon M \to \mathbb{R}^{n+k}$ of a manifold M^n. For $p, q \in M$, set $d(p,q) := |f(p) - f(q)|$. Is d a metric?

b) As before, but for the case that f is an embedding. Prove d defines a metric on M.

Problem 13 – Immersions and embeddings:

a) The improved form of the Whitney embedding theorem says that each n-manifold can be immersed into $\mathbb{R}^{2n-1}$, and embedded into $\mathbb{R}^{2n}$. Discuss these statements for the case $n = 1$.

b) Can a Möbius strip be embedded into $\mathbb{R}^3$?

c) Find a 2-manifold which cannot be embedded into $\mathbb{R}^3$ (and reason for this fact).

Problem 14 – Preparation for Thursday's class:

a) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Define the directional derivative of f at $p \in \mathbb{R}^m$ with respect to a direction $\xi \in \mathbb{R}^m$.

b) Relate the directional derivative to the differential; state the result also with sums and indices, avoiding matrix notation.

Problem 15 – Tangent space:

a) How did we define a tangent vector $v \in T_pM$ to a manifold M? What is the standard basis of T_pM with respect to a chart (x, U)?

b) Consider an implicitely defined submanifold $M = \varphi^{-1}(0)$, where φ has 0 as a regular value. How can you describe the tangent space?

c) If y is a chart which locally maps a submanifold $M \subset \mathbb{R}^{n+k}$ to a slice, i.e. $y(M \cap U) = y(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^k$, where $U \subset \mathbb{R}^{n+k}$, how would you describe the tangent space of M at $p \in M$?

Problem 16 – Grassmannians:
We consider

$$G(k,n) := \{k - \text{dimensional subvectorspaces } V \subset \mathbb{R}^n\}.$$

We want to prove that $G(k,n)$ is a manifold with a suitable differentiable structure.

a) Consider $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Prove that $U := \{V \in G(k,n) : V \cap (\{0\} \times \mathbb{R}^{n-k} = \{0\}\}$ is a manifold by regarding U as the set of graphs $\Gamma(A)$ of linear mappings $A\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$. What is the dimension? (Perhaps the same works implicitely.)

b) Find charts on sets similar to U that cover $G(k,n)$.

c) Show that the transition maps are differentiable (this is harder).

d) Find a bijection from $G(k,n)$ to $G(n-k,n)$. Is it a diffeomorphism?