Fachbereich Mathematik
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17.11.2009

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3. Problems for Manifolds

Problem 8 – Continuous image of a set of measure 0 with positive measure:
Let $Q := [0,1] \times [0,1]$ be the square in the plane $\mathbb{R}^2$. A space-filling curve is a curve $c\colon [0,1] \to Q$ which is continuous and surjective. Use this example to construct a continuous mapping $f\colon Q \to Q$ which maps a set of measure 0 to a set of positive measure.

Problem 9 – Matrices of fixed rank:

a) Check that the space of $m \times n$-matrices with rank 1 has the dimension stated in class. Is there any other rank, besides 0 and $\min(m,n)$, with an obvious dimension?
b) How many charts for M_r have we used in class to describe this submanifold of M?
c) Let $K \subset \mathbb{R}^n$ be compact, and $f\colon K \to \mathbb{R}^{n+k}$ with $k \geq 0$ be any differentiable map, i.e., f extends as a differentiable map to some open neighbourhood U of K. Find a matrix A, such that $x \mapsto Ax + f(x)$ has a Jacobian of rank n for all $x \in K$.

Problem 10 – Helicoids in $\mathbb{S}^3$:
Let $a \in \mathbb{R}$ be a parameter and consider the mapping

$$h = h_a\colon \mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4, \qquad (u,v) \mapsto \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \cos(av) \\ \sin u \sin(av) \end{pmatrix}.$$

a) h_a is an immersion of $\mathbb{R}^2$ for $a \neq 0$.
Hint: Calculate the determinant of a 2×2 minor of the Jacobian $Jh = \left(\frac{\partial}{\partial u}h, \frac{\partial}{\partial v}h\right)$.
b) Show the two axes $v \mapsto a_1(v) = h(0,v)$ and $v \mapsto a_2(v) = h(\frac{\pi}{2}, v)$ are great circles whose points are pairwise perpendicular. Identifying $\mathbb{R}^4$ with $\mathbb{C} \times \mathbb{C}$, how would you write a_1 and a_2?
c) The maps $u \mapsto h(u,v) = (\cos u)a_1(v) + (\sin u)a_2(v)$ parameterize great circles with unit speed, and these circles meet the two axes at right angles. (What does it mean for two curves to meet at a right angle?) In this sense, h represents a helicoid in $\mathbb{S}^3$.
d) Try to identify the image surface for $a = 0$. What is the set where h_0 fails to be an immersion and what is its image?
e) Consider $a = 1$. What is the speed $\|\frac{\partial}{\partial v}h\|$? Find *periods* for h, that is, $(c,d) \neq (0,0)$ with minimal length such that $h(u+c, v+d) = h(u,v)$. Use a (non-rigorous) orientability argument to determine the topological type of the image surface.

Problem 11 – Klein bottle:
We use the following which has not been defined formally in class: The Klein bottle is a non-orientable manifold, obtained by identifying opposite edges of a square: One pair of opposite edges in the same direction, the other in opposite directions.

a) Reason geometrically why the Klein bottle cannot be embedded into $\mathbb{R}^3$.
Hint: An embedding defines a continous normal.
b) Prove that the helicoid h_2 (or $h_{1/2}$) represents a Klein bottle immersed in $\mathbb{R}^4$. To do so, determine again minimal *periods* for h as in the previous problem.
c) Does h_2 represent an embedding of the Klein bottle into $\mathbb{S}^3$?