Fachbereich Mathematik
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3.11.2009

TECHNISCHE
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2. Problems for Manifolds

Problem 4 – Tangent vectors to $\mathbb{S}^2$:

The following curves in $\mathbb{S}^2$ are defined in a neighbourhood of $t = 0$. Which curves are equivalent in $\mathbb{S}^2$ and define the same tangent vector?

$$c_1(t) = \big(\cos t, 0, \sin t\big) \qquad c_2(t) = \big(\sin t, 0, \cos t\big),$$
$$c_3(t) = \big(\cos(2t), 0, \sin(2t)\big), \qquad c_4(t) = \big(\sqrt{1-t^2}, 0, t\big)$$

Check first that $c_i(0)$ agrees, and then for a chart x that $(x \circ c)'(0)$ agrees. A good choice of x is projection to a coordinate plane (verify that x a chart!).

Problem 5 – Tangent vectors to $\mathbb{R}P^2$:

Consider the point $p = [1, 1, 0] \in \mathbb{R}P^2$ and the charts x_1 and x_2 given in the lecture.

a) Find curves $c_1(t)$, $c_2(t)$ in $\mathbb{R}^2$ which represent the standard basis at p w.r.t. x_1.

b) Decide if c_1, c_2 also represent the standard basis w.r.t. x_2. To do so, consider the representing curves $d_i(t) := (x_2 \circ x_1^{-1})(c_i(t))$ in the image of x_2.

c) Which linear mapping maps $c_i'(0)$ to $d_i'(0)$?

Problem 6 – Minimal atlas:

Let M be a compact manifold, containing at least two points. Show that each atlas of M contains at least two charts. In particular the stereographic atlas of $\mathbb{S}^n$ is minimal.

Problem 7 – Vector fields and division algebras:

Assume that on some $\mathbb{R}^n$ there is the structure of a *division algebra*, that is, a bilinear map $\beta\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, written as $(x, y) \mapsto xy$, such that all maps

$$\lambda_x\colon \mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto xy \qquad \text{and} \qquad \rho_y\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto xy$$

are bijective. We do not assume that the multiplication β is associative, but we assume there is a unit element $e \in \mathbb{R}^n$ with $ex = xe = x$ for all $x \in \mathbb{R}^n$. Prove the following:

a) If $n > 1$ and $x \notin \mathbb{R}e$ then λ_x has no real eigenvalues.
Hint: If $xy = \mu y$ then $(x - \mu e)y = 0$.

b) n is even. *Hint:* Recall a linear algebra result on eigenvalues.

c) We extend $b_n = e$ to a basis $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ and consider the corresponding vector fields $X_j := X_{\lambda_{b_j}}$ for $j = 1, \dots, n$ on $\mathbb{S}^{n-1}$. Show that for each $x \in \mathbb{S}^{n-1}$, the vectors $X_1(x), \dots, X_{n-1}(x)$ are linearly independent.
Hint: $\operatorname{span}\{x, b_1x, \dots, b_{n-1}x\} = \rho_x(\mathbb{R}^n) = \mathbb{R}^n$.

d) An n manifold is *parallelizable* if there are n vector fields which give a basis of each tangent space. Show that $\mathbb{S}^{n-1}$ is parallelizable, if $\mathbb{R}^n$ carries the structure of a division algebra.

e) Show that the matrix group

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

gives $\mathbb{R}^4 = \mathbb{C}^2$ the structure of a four dimensional associative division algebra, called