MANIFOLDS
WS 09/10

KARSTEN GROSSE-BRAUCKMANN

Contents

References

(*i*) Lee [Lee]: Introduction to smooth manifolds, Springer 2003 (comprehensive, quite explicit)

(*ii*) Boothby [B]: An introduction to differentiable manifolds and Riemannian geometry, Academic Press 1975 (A classic, also explicit.)

(*iii*) Warner [W]: Foundations of differentiable manifolds and Lie groups, Springer 1983 (clear and short)

(*iv*) Spivak [Spi]: Differential Geometry, vol 1, Publish or Perish 1979

Introduction

The class addressed students from the third year on. I tried to give life to the somewhat formal topic of manifolds by aiming at some concrete theorems. In a two-hour class such as this one, a focus on some particular aspects is appropriate anyway.

The first goal of the class is a version of the Whitney embedding theorem. It says that a given abstract manifold can be realized as a submanifold of Euclidean space with twice the dimension.

The second theorem presented is the Frobenius integrability theorem. Given a distribution of lines or planes, etc. it decides if there is a curve or surface, etc. such that the distribution is tangent to it. While for a line field, there is no problem, in the higher dimensional cases there is a necessary and sufficient condition in terms of the commutator of two vector fields tangent to the distribution.

The third and last theorem I covered is Stokes' theorem. It generalizes the fundamental theorem of calculus to a form which includes all classical integral theorems, such as the divergence theorem. Stokes' theorem requires the machinery of differential forms. While I had to be a bit sketchy in class eventually, these notes should be essentially complete. It was sad to realize I could not cover any of the applications.

The problems presented in seven sessions are also included.

I thank Dominik Kremer for communicating many corrections to these notes.

Darmstadt, 16. February 2010 and Sept. 10

Part 1. Differentiable manifolds and the Whitney embedding theorem

1. Lecture, Thursday 15.10.09

1. The definition of a differentiable manifold

Differentiable manifolds are the abstract generalization of the notion of submanifolds which are in turn generalizations of curves and surfaces. A submanifold of dimension n is a subset $M \subset \mathbb{R}^{n+k}$ which has three equivalent descriptions in the neighbourhood $U \subset M$ of each point:
Implicit: The inverse image $U := \varphi^{-1}(b) \subset \mathbb{R}^{n+k}$: of a regular value b of a function $\varphi \in C^\infty(\mathbb{R}^{n+k}, \mathbb{R}^k)$, that is $d\varphi|_M$ has rank k.
Parameterized: Local image of a parameterization $f \in C^\infty(V \subset \mathbb{R}^n, \mathbb{R}^{n+k})$, where that is, has rank n on V.
Graphs: Local parameterizations of the form $U = \{(x, g(x)) : x \in D \subset \mathbb{R}^n, g\colon C^\infty(D, \mathbb{R}^k)\}$

These definitions imply that submanifolds are smooth and locally look like deformations of $\mathbb{R}^n$. They neither have self-intersections nor boundary. They may, however, have several connected components. Examples to keep in mind are the spheres $\mathbb{S}^n$ or more generally quadrics, and matrix groups like $\mathsf{O}(n)$ or $\mathsf{GL}(n)$.

While these examples are naturally given as subsets of $\mathbb{R}^{n+k}$ there are many other cases where spaces arise without an ambient space. Configuration spaces form an example: For instance, the space of m pairwise distinct points on the sphere $\mathbb{S}^2$, or the space of polygons in $\mathbb{R}^2$. Many more complicated examples like DNA strings are given similarly. Quotient constructions are a very natural way to construct manifolds, as we will see; their construction does not yield a containing ambient space.

Often manifolds arise with additional structure: Riemannian manifolds, Lie groups, symplectic manifolds, Kähler manifolds, Poisson manifolds, etc.

Historically, Riemann presented an intuitive notion of a manifold in his inaugural lecture from 1853, using foundational ideas of Gauss. The formal notion of a manifold was given by Hermann Weyl, and is contained in his book *Die Idee der Riemannschen Fläche* from 1913, for the case of complex surfaces. These ideas became important for the theory of general relativity, developed at the time.

1.1. **Topological manifolds.** Submanifolds are given as subsets of some $\mathbb{R}^m$. We will define manifolds abstractly and so we need to say which kind of space we work with.

We depend on the basic notions of topology. M is a *topological space* if there is a system of sets $\mathcal{O}$, called *open sets*, such that:

• arbitrary unions und finite intersections of sets in $\mathcal{O}$ are in $\mathcal{O}$,
• the empty set and M belong to $\mathcal{O}$.
Having said this, we can define a map $f\colon M \to N$ between topological spaces to be *continuous* by requiring that open sets of N have open sets of M as preimages.

An important example are metric spaces. By definition, a subset $U \subset M$ is an element of $\mathcal{O}$, if each point $p \in M$ has a distance ball $\{q \in M : d(p,q) < r(p)\}$ which is contained in U. (Please verify the above properties.)

We will demand the following properties of our topological space M:

• M is *Hausdorff* if for any pair of points $p, q \in M$ there are two open sets $U, V \in \mathcal{O}$ with $p \in U$, $q \in V$, which are disjoint, $U \cap V = \emptyset$.

• M is *second countable* if there is a countable basis for the topology. Here, a *basis* is a family of sets $\mathcal{B}$, such that $\bigcup\{B \in \mathcal{B}\} = M$, and such that given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$. If there is a countable such family, this means that there are not too many open sets. In $\mathbb{R}^n$, for example, we could take for $\mathcal{B}$ the balls of rational radius centered at points with rational coordinates. Second countability will be signifcant when it comes to constructing partitions of unity.

• M is *locally Euclidean of dimension n* if each point of M has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$, that is, for all $p \in M$ there exist open sets $U \subset M$, $\Omega \subset \mathbb{R}^n$ and a homeomorphism $x\colon U \to \Omega$. It will be convenient to assume that U is connected. Then x is called a *chart* of M. Our definition means that each chart respects the given topologies of $\mathbb{R}^n$ and M.

Definition. A *topological manifold of dimension* $n \in \mathbb{N}$ is a topological space M which is Hausdorff, second countable and locally Euclidean of dimension n.

Examples. 1. The only zero-dimensional manifolds are finite or countable unions of points.
2. All one-dimensional connected manifolds are homeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. See Guillemin/Pollack, appendix.
3. Graphs of continuous functions over open sets, for instance a cone in $\mathbb{R}^3$ (or $\mathbb{R}^n$).
4. A double cone in $\mathbb{R}^3$ is not a manifold since it is not locally Euclidean at 0.

Remark. All our manifolds will turn out to be metric spaces. These spaces are always Hausdorff. Although they are not necessarily second countable, the consequence of it we need, paracompactness, is always satisfied; see Munkres, *Topology*, Ch. 6, Thm. 41.4. Thus in fact we need not bother about these two properties.

1.2. **Differentiable manifolds.** By requiring that our charts be homeomorphisms we endow M with the topology of the parameterizing subsets of $\mathbb{R}^n$. This very idea is useful to define the differentiability of manifolds.

Note that the *transition map* of two charts of a topological manifold,

$$y \circ x^{-1}\colon \quad x(U \cap V) \to y(U \cap V), \tag{1}$$

is a homeomorphism of the appropriate subsets of $\mathbb{R}^n$.

Later we will define items such as a differentiable map f on a manifold by demanding that its composition with a chart $f \circ x^{-1}$ be differentiable. In order to do so, we will need that this definition is independent of the particular chart chosen, that is, we need that (1) is differentiable.

We say charts (x, U), (y, V) are *differentiably compatible* if (1) and its inverse is differentiable (C^∞), or a diffeomorphism. This will hold in particular if $U \cap V = \emptyset$.

A set of charts $\mathcal{A} = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ with $\bigcup_{\alpha\in A} U_\alpha = M$ is called an *atlas* of M. An atlas $\mathcal{A}$ is a *differentiable atlas* if all charts $(x, U) \in \mathcal{A}$ are *differentiably compatible.*

Obviously, there is no harm in adding charts to an atlas, as long as they are *differentiably compatible with* $\mathcal{A}$, that is, compatible with each chart of $\mathcal{A}$.

The following useful definition avoids refering to the notion of equivalence classes of atlasses:

Definition. A *differentiable structure* on a topological manifold is a set of charts $\mathcal{S}$, called *maximal differentiable atlas*, such that there exists an atlas $\mathcal{A} \subset \mathcal{S}$, and $\mathcal{S}$ contains all charts that are differentiably compatible with $\mathcal{A}$.

Example. 1. If $\mathcal{A} = \{(\mathrm{id}, \mathbb{R}^n)\}$ then

$$\mathcal{S} = \{(f, U) : U \subset \mathbb{R}^n, f\colon U \subset \mathbb{R}^n \to \mathbb{R}^n \text{ diffeomorphism onto its image}\}.$$

2. An intuitive notion of differentiable structure is that it tells us which subsets are straight and which ones we consider to have corners. To see this, consider two different differentiable structures on $\mathbb{R}^n$: $\mathcal{A}_1 := \{(\mathbb{R}^n, \mathrm{id})\}$, $\mathcal{A}_2 := \{(\mathbb{R}^n, f)\}$, where f is 1-homogeneous, preserves rays through the origin as sets, and maps the unit cube onto the unit ball. For the case $n = 2$, in the second structure a square (centered at the origin) is a nice differentiable object, while a circle is not. This can be made precise once differentiable curves are available.

Proposition 1. *A differentiable structure $\mathcal{S}$ is itself a differentiable atlas, and so there is a unique maximal atlas containing $\mathcal{S} \supset \mathcal{A}$.*

Proof. We show only the first statement. Let $(x, U), (y, V) \in \mathcal{S}$ be differentiably compatible to the atlas $\mathcal{A}$, then they are differentiably compatible charts.

For each point $p \in U \cap V$ the atlas $\mathcal{A}$ contains a chart (x_α, U_α) containing p. Then at p

$$x \circ y^{-1} = \underbrace{\underbrace{(x \circ x_\alpha^{-1})}_{\text{differentiable, since compatible to } \mathcal{A}} \circ \underbrace{(x_\alpha \circ y^{-1})}_{y \text{ compatible to } \mathcal{A}}}_{\text{differentiable by chain rule}}$$

$\square$

Definition. A (*differentiable*) *manifold* M is a pair $(M, \mathcal{S})$ where M is a connected topological manifold M and $\mathcal{S}$ a differentiable structure. If $\mathcal{A} \subset \mathcal{S}$ is an atlas we will also call $(M, \mathcal{A})$ or M a manifold and say *chart* for a chart of $\mathcal{A}$.

Whenever we say *differentiable* we mean smooth or C^∞. We could define similarly C^k-manifolds or analytic (C^ω) manifolds, by requiring the transition maps are in these classes.

`2. Lecture, Thursday 22.10.09` ______________________________

1.3. **Examples of differentiable manifolds.** 1. Any connected open subset U of a manifold M is a manifold itself. Then $\mathcal{O} = \{\text{open subsets of } U\}$ is called the *subspace topology* or *relative topology*. The structure $\mathcal{S}$ is given by the maps (x, U) of the differentiable structure of M, with $U \subset \mathcal{O}$.

2. $\mathbb{R}^n$ is a differentiable manifold with the atlas $(\mathrm{id}, \mathbb{R}^n)$.

3. The structures $(\mathrm{id}, \mathbb{R})$ and $(x^3, \mathbb{R})$ are different. Similarly, distinct differentiable structures on $\mathbb{R}^n$ arise from a single chart which is a homeomorphism but not a diffeomorphism.

4. However, there are differentiable structures on $\mathbb{R}^4$ which are not homeomorphism equivalent to the standard structure, so-called *exotic structures*. The same holds for spheres in most dimensions.

5. *Spheres* $\mathbb{S}^n := \{p \in \mathbb{R}^{n+1} : p_1^2 + \ldots + p_{n+1}^2 = 1\}$.

We use *stereographic projection* onto the equatorial plane to define two charts. Let $N := (0, \ldots, 0, 1)$ be the north pole and $-N$ the south pole. Given $p \in \mathbb{S}^n$, we determine $\lambda \neq 1$, such that a point on the straight line through p and $\pm N$ has last coordinate 0:

$$\big(x_\pm(p), 0\big) \overset{!}{=} \lambda p \pm (1 - \lambda) N$$

The first n coordinates give

$$x_\pm(p) = \lambda(p_1, \ldots, p_n),$$

while the last coordinate determines λ:

$$0 = \pm(1 - \lambda) + \lambda p_{n+1} = \pm 1 + \lambda(\mp 1 + p_{n+1}) \quad \Rightarrow \quad \lambda = \frac{\mp 1}{\mp 1 + p_{n+1}} = \frac{1}{1 \mp p_{n+1}}$$

Thus we define

$$x_\pm \colon U_\pm := \mathbb{S}^n \setminus N_\pm \to \mathbb{R}^n, \qquad x_\pm(p) := \frac{1}{1 \mp p_{n+1}} (p_1, \ldots, p_n).$$

Now we claim: $\mathcal{A} := \big\{(x_+, U_+), (x_-, U_-)\big\}$ is an atlas. Clearly, $U_\pm$ cover the entire sphere $\mathbb{S}^n$. To see they are homeomorphisms, we claim the following maps are the inverses:

$$x_\pm^{-1}\colon \mathbb{R}^n \to U_\pm, \quad x_\pm^{-1}(u) := \left(\frac{2u}{|u|^2+1},\ \pm(1 - \frac{2}{|u|^2+1})\right) = \frac{1}{|u|^2+1}\Big(2u,\ \pm(|u|^2-1)\Big).$$

Indeed, for all $u \in \mathbb{R}^n$

$$\begin{aligned} x_\pm\big(x_\pm^{-1}(u)\big) &= x_\pm\left(\frac{2u}{|u|^2+1},\ \pm(1-\frac{2}{|u|^2+1})\right) \\ &= \frac{1}{1-\big(1-\frac{2}{|u|^2+1}\big)}\cdot\frac{2u}{|u|^2+1} = \frac{1}{\frac{2}{|u|^2+1}}\cdot\frac{2u}{|u|^2+1} = u. \end{aligned}$$

Since $x_\pm$ and $x_\pm^{-1}$ can be seen to be bijective this suffices to prove that $x_\pm^{-1}$ is indeed the inverse of $x_\pm$.

Clearly, our maps are continuous with respect to the submanifold topology of $\mathbb{S}^n$. It remains to study the two transition maps

$$\begin{aligned} \big(x_\pm \circ x_\mp^{-1}\big)(u) &= x_\pm^{-1}\left(\frac{2u}{|u|^2+1},\ \mp(1-\frac{2}{|u|^2+1})\right) \\ &= \frac{1}{1+(1-\frac{2}{|u|^2+1})}\cdot\frac{2u}{|u|^2+1} = \frac{1}{2-\frac{2}{|u|^2+1}}\cdot\frac{2u}{|u|^2+1} = \frac{u}{|u|^2}. \end{aligned} \tag{2}$$

Both are defined on $x_\mp(U_+\cap U_-) = x_\mp\big(\mathbb{S}^n \setminus \{N, -N\}\big) = \mathbb{R}^n \setminus \{0\}$ and are indeed differentiable. Geometrically they represent an inversion in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$.

A simpler choice of charts is given by the $2n$ hemispheres $H_\pm^k := \{p \in \mathbb{S}^n : \langle p, e_k\rangle > 0\}$, which cover $\mathbb{S}^n$. Each hemisphere can be represented as a graph over some coordinate hyperplane. However, stereographic projection is not only nicer in that two charts are sufficient, but it has a useful additional property: It is conformal, that is, angle preserving. This property also holds for the inversion map.

Problem. Check how our calculations change when $\mathbb{S}^n$ is replaced by the sphere $\mathbb{S}^n_r$ of radius $r > 0$.

6. *Projective spaces* $\mathbb{K}P^n$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; here $\mathbb{H}$ denotes quaternions.

These are the sets of $\mathbb{K}$-lines $\mathbb{K}^{n+1}$ with the following differentiable structure. The relation $u \sim \lambda u$ for some $\lambda \in \mathbb{K}$ is an equivalence relation on $\mathbb{K}^{n+1}\setminus\{0\}$. Let $u = (u_1, \dots, u_{n+1}) \in \mathbb{K}^{n+1}\setminus\{0\}$ and $[u]$ be its class.

For $i = 1, \dots, n+1$ let us define *homogeneous coordinates* on the spaces $\mathbb{K}^n = \mathbb{R}^n$, $\mathbb{R}^{2n}$, or $\mathbb{R}^{4n}$:

$$x_i\colon U_i := \big\{[u] : u_i \neq 0\big\} \subset \mathbb{K}P^n \to \mathbb{K}^n, \qquad x_i\big([u]\big) = \frac{1}{u_i}(u_1, \dots, \widehat{u_i}, \dots, u_{n+1})$$

Note that on the affine hyperplane $H_i := \{u \in \mathbb{K}^{n+1} : u_i = 1\}$, the map x_i is the identity, and all scalar multiples of points $p \in H_i$ are clearly mapped onto the same point. The charts x_i to induce a topology on $\mathbb{K}P^n$.

We claim that the n charts $\mathcal{A} := \{(x_i, U_i) : i = 1, \dots, n\}$ form an atlas. First, x_i is injective. Moreover,

$$x_i^{-1} \colon \mathbb{K}^n \to U_i \qquad x_i^{-1}(u_1, \dots, u_n) := [u_1, \dots, u_{i-1}, 1, u_i, \dots, u_n]$$

is the inverse of x_i, as for all $u \in \mathbb{K}^n$:

$$x_i\big(x_i^{-1}(u)\big) = x_i\big([u_1, \dots, u_{i-1}, 1, u_i, \dots, u_n]\big) = \frac{1}{1}(u_1, \dots, u_{i-1}, u_i, \dots, u_n) = u.$$

Let us now show differentiability of the transition maps, first for $j < i$:

$$\big(x_j \circ x_i^{-1}\big)(u) = x_j\big([u_1, \dots, u_{i-1}, 1, u_i, \dots, u_n]\big) = \frac{1}{u_j}(u_1, \dots, \widehat{u_j}, \dots u_{i-1}, 1, u_i, \dots, u_n),$$

where

$$u \in x_i\big(U_i \cap U_j\big) = \{u \in \mathbb{K}^n : u_i \neq 0 \text{ and } u_j \neq 0\}.$$

Similarly for $j > i$. This proves differentiability of the transition maps.

Problem. Think this through for $\mathbb{R}P^2$. What are the maps geometrically? What is the exceptional set $U_i \subset \mathbb{R}P^2$? Can you relate $\mathcal{A}$ to the atlas of hemispheres for $\mathbb{S}^n$?

7. *Grassmannians [Grassmann-Räume]* $G(k, n)$ are the sets of k-dimensional subspaces of $\mathbb{R}^n$. In case $k = 1$ they agree with real projective space, $G(1, n) = \mathbb{R}P^{n-1}$. Taking the (unoriented) normal of a hyperplane, we see that $G(n-1, n) = \mathbb{R}P^{n-1}$ as well. These spaces are easy to describe as quotient spaces, but expicit coordinates are somewhat tedious.

8. *Lie groups* are manifolds which are groups. Typical examples are $\mathsf{GL}_+(n, \mathbb{R})$, $\mathsf{SO}(n, \mathbb{R})$, $\mathsf{SL}(n, \mathbb{R})$. Note that all of $\mathsf{GL}(n, \mathbb{R})$ or $\mathsf{O}(n, \mathbb{R})$ has two connected components, given by the matrices with positive or negative determinant. Other Lie groups are the tori $T^n = \mathbb{R}^n/\mathbb{Z}^n$. The only spheres which are Lie groups are $\mathbb{S}^1$ and $\mathbb{S}^3$; the group structure for the latter is given by the unit quaternions.

9. Other constructions of manifolds involve products or quotients. Also, regular values of functions define level sets which are manifolds.

1.4. **Differentiable maps.** We define differentiability of mappings between manifolds by requiring that their composition with charts be differentiable:

Definition. Let M and N be (differentiable) manifolds. Then $f \colon M \to N$ is *differentiable* at $p \in M$ if $y \circ f \circ x^{-1}$ is differentiable at p, where (x, U) is a chart at p and (y, V) a chart at $f(p)$.

Our definition is independent of the particular charts chosen: With respect to other charts $\tilde{x}$ at p and $\tilde{y}$ at $f(p)$ we can write on suitable domains

$$\tilde{y} \circ f \circ \tilde{x}^{-1} = (\tilde{y} \circ y^{-1}) \circ (y \circ f \circ x^{-1}) \circ (x \circ \tilde{x}^{-1}).$$

The composition on the right hand side is in terms of two transition maps which are differentiable. Hence, by the chain rule, $\tilde{y} \circ f \circ \tilde{x}^{-1}$ is differentiable if and only if $(y \circ f \circ x^{-1})$ is.

By the same token differentiability is preserved under composition: To see this we write $z^{-1} \circ f \circ g \circ x = (z^{-1} \circ f \circ y) \circ (y^{-1} \circ g \circ x)$ and apply the chain rule once again.

Examples. 1. Trivially, the identity on M is differentiable since transition maps are differentiable.
2. We will always consider $\mathbb{R}^n$ with the differentiable structure given by the atlas $(\mathbb{R}^n, \mathrm{id})$. This makes each chart $x_\alpha \colon U_\alpha \to \mathbb{R}^n$ of a manifold M into a differentiable mapping; indeed, $\mathrm{id} \circ x_\alpha \circ x_\beta^{-1}$ is a transition map and hence differentiable. For instance, stereographic projection $x_\pm$ is a differentiable mapping from the manifold $\mathbb{S}^n \setminus \{\pm N\}$ to $\mathbb{R}^n$.

Definition. A *diffeomorphism* $f \colon M \to N$ between manifolds is a homeomorphism such that f and f^{-1} are differentiable. Then we call M and N *diffeomorphic (manifolds)*.

Note that if M is diffeomorphic to N then $\dim M = \dim N$ (why?).

Examples. 1. $\mathbb{R}^n$ and $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ are diffeomorphic via $x \mapsto \frac{x}{|x|} \operatorname{arctanh} |x|$.
2. T^2 and the torus of revolution are diffeomorphic (via?).
3. If x_α is a chart then x_α^{-1} is differentiable. Indeed, $x_\beta \circ x_\alpha^{-1} \circ \mathrm{id}^{-1}$ is a transition map and so differentiable. Thus each chart is a diffeomorphism onto its image.

`3. Lecture, Thursday 29.10.09` ________________________________

2. Tangent space

In contrast to a topological manifold, a differentiable manifold has a tangent space, which crucial in the for all results we will discuss later

2.1. **Equivalence classes of curves.** For the case of submanifolds of $\mathbb{R}^n$, a tangent space is represented by the set of tangent vectors to curves. Remember the case $\mathbb{S}^n$: A differentiable curve c in $\mathbb{S}^n$ with $c(0) = p$ satisfies $|c|^2 \equiv 1$ and so $\frac{d}{dt}|c|^2(0) = 2\langle c'(0), p\rangle = 0$. Conversely, each $v \perp p$ is the tangent vector of the curve $c(t) = \cos(t|v|)p + \sin(t|v|)\frac{v}{|v|}$. Thus $T_p\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} \colon \langle v, p\rangle = 0\}$.

Similarly, we wish to represent the tangent space of a manifold at a point p by the set of velocity vectors of curves through p. We need to give the definition in terms of charts, and will introduce tangent vectors as equivalence classes of curves which a chart maps to the same tangent vector in $\mathbb{R}^n$.

Definition. (*i*) A (differentiable) *curve* c on a manifold M is a differentiable map $c\colon I \to M$, where I is an open interval. We say c is a *curve through* $p \in M$ if $0 \in I$ and $c(0) = p$.
(*ii*) A *tangent vector* to M^n at $p \in M$ is an equivalence class of curves through p under the following relation:

$$c_1 \sim c_2 \quad :\Longleftrightarrow \quad \exists \text{ chart } x \text{ at } p\colon\ \frac{d}{dt}(x \circ c_1)\Big|_0 = \frac{d}{dt}(x \circ c_2)\Big|_0 \in \mathbb{R}^n$$

We denote the set of tangent vectors through $p \in M$ with $T_pM = \{[c]\colon c(0) = p\}$.

This definition means that two curves c_1 and c_2 represent the same tangent vector if first $c_1(0) = c_2(0)$ and second for a given chart they have the same euclidean tangent vectors at time $t = 0$.

To see the relation is independent of the chart x chosen, take another chart y at p:

$$\text{(3)}\quad \text{For } i = 1, 2: \quad \frac{d}{dt}(y \circ c_i)\Big|_0 = \frac{d}{dt}(y \circ x^{-1} \circ x \circ c_i)\Big|_0 \overset{\text{chain rule}}{=} \underbrace{d(y \circ x^{-1})_{x(p)}}_{\text{independent of } i=1,2} \cdot \frac{d}{dt}(x \circ c_i)\Big|_0$$

Let $p \in M^n$. We call $\xi = \frac{d}{dt}(x \circ c)|_{t=0} \in \mathbb{R}^n$ the *principal part* of the tangent vector $[c] \in T_pM$ with respect to the chart x. We can read (3) to say:

Theorem 2 (Transformation rule for tangent vectors)**.** *Let x and y be charts at p. For $v \in T_pM$ let ξ or η be the principal parts with respect to x or y, respectively. Then*

$$\eta = d(y \circ x^{-1})_{x(p)}\, \xi. \tag{4}$$

Thus we can say: Tangent vectors transform with the Jacobian of the transition map.

Example. In the sphere $M := \mathbb{S}^2$, consider the longitude $c(t) = (\cos t, 0, \sin t)$ through $p := c(0) = (1, 0, 0)$. With respect to the charts $x_\pm$ let us compute the principal parts:

$$x_\pm \circ c = \left(\frac{\cos t}{1 \mp \sin t}, 0\right) \quad \Rightarrow \quad \frac{d}{dt}(x_\pm \circ c)\Big|_{t=0} = \left(\frac{0 - (\mp 1)}{1}, 0\right) = (\pm 1, 0) =: \xi_\pm \tag{5}$$

The transformation rule now says $\xi_- = d(x_- \circ x_+^{-1})\xi_+$. But the transition map (2) is inversion in the unit circle, and so its linearisation is a reflection in the y-line tangent to the circle, agreeing with our result.

Using matrices, (4) becomes

$$\eta^j = \sum_{i=1}^n \frac{\partial (y \circ x^{-1})^j}{\partial u^i}(p)\, \xi^i, \qquad j = 1, \ldots, n.$$

It is common to write $\frac{\partial y^j}{\partial x^i}(p) := \partial_i(y \circ x^{-1})^j$ so that the transformation rule becomes $\eta^j = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i}\xi^i$, just like the chain rule in Euclidean space.

Since $\xi = \frac{d}{dt}(u + t\xi)$ the curves $c(t) = x^{-1}(u + t\xi)$ represent tangent vectors in a 1-1 way. Thus we can consider the classes

$$\mathbb{R}^n \to T_pM, \qquad \xi \mapsto [c(t) := x^{-1}(x(p) + t\xi)] \tag{6}$$

This map defines a vector space structure on T_pM:

Theorem 3. *The set T_pM has the structure of an n-dimensional vector space such that* (6) *becomes an isomorphism regardless of the chart chosen.*

Proof. By (6), we can define addition and scalar multiplication with respect to a chart (x, U) by setting

$$\lambda\big[x^{-1}(x(p) + t\xi)\big] + \big[x^{-1}(x(p) + t\eta)\big] = \big[x^{-1}(x(p) + t(\lambda\xi + \eta))\big].$$

This is independent of the chart chosen since for another chart (y, V) the tangent vectors transform under the linear map (4). $\square$

The vector space isomorphism (6) maps the standard basis $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ to a basis

$$(e_1, \dots, e_n) := \Big([x^{-1}(x(p) + tb_1)],\ \dots\ , [x^{-1}(x(p) + tb_n)]\Big)$$

of T_pM. We call $e_i = e_i(p)$ the *standard basis of T_pM with respect to the chart x*. Each $v \in T_pM$ is a linear combination

$$v = \big[x^{-1}(x(p) + t\xi)\big] = \Big[x^{-1}\big(x(p) + t\sum \xi^i b_i\big)\Big] = \sum \xi^i\big[x^{-1}(x(p) + tb_i)\big] = \sum \xi^i e_i \tag{7}$$

In general, another chart (y, V) at p will lead to a different standard basis; only in case $d(y \circ x^{-1})_{x(p)} = \mathrm{id}$, that is, for $\frac{\partial y^j}{\partial x^i}(p) = \delta_i^j$, the two bases agree. We say that T_pM does not have a *canonical basis.*

Examples. 1. Again we consider $p := (1, 0, 0) \in M := \mathbb{S}^2$. For x_+, the standard basis at p is $e_1 = (0, 0, 1)$, $e_2 = (0, 1, 0)$, while for x_- the standard basis at p is $e_1 = (0, 0, -1)$, $e_2 = (0, 1, 0)$. Indeed, the first tangent vector was computed in (5), while the second is immediate since both charts map the equatorial unit circle of $\mathbb{S}^2$ to the unit circle in the plane, that is, $x_\pm(\cos t, \sin t, 0) = (\cos t, \sin t)$.

2. On $\mathbb{R}^n$ we always use the atlas $\{\mathrm{id}, \mathbb{R}^n\}$. Then the isomorphism (6) becomes a *canonical isomorphism.* We will not distinguish between principal parts and equivalence classes of curves, and so we identify

$$[p + t\xi] = \xi. \tag{8}$$

Here, we equate a representing curve with a vector, namely its tangent! Using this identification we can say $T_p\mathbb{R}^n = \mathbb{R}^n$.

2.2. **The tangent bundle TM.** We need a unified treatment of the vector spaces T_pM together with their base points p. This will later allow us to define the differentiability of vector fields. A good picture to keep in mind is for $M = \mathbb{S}^1$: Then the set of base points with tangent vectors TM is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. To describe TM we use the product of charts for M with $\mathbb{R}^n$, where the latter factor is represented by principal parts.

To do so, let (x_α, U_α) be a chart, and $[c] \in T_pM$ be a tangent vector. Then we define

$$y_\alpha\colon \bigcup_{p\in U_\alpha} T_pM \to x_\alpha(U_\alpha) \times \mathbb{R}^n \subset \mathbb{R}^{2n}, \qquad y_\alpha([c]) := \Big(x_\alpha(c(0)), \frac{d}{dt}(x_\alpha \circ c)|_{t=0}\Big), \tag{9}$$

Theorem 4. *Let M^n be a manifold with atlas $\mathcal{A}_M = \{(x_\alpha, U_\alpha) : \alpha \in A\}$. Then the disjoint union $TM := \bigcup_{p\in M} T_pM$ becomes a $2n$-dimensional manifold, by defining the atlas*

$$\mathcal{A}_{TM} := \Big\{(y_\alpha, \bigcup_{p\in U_\alpha} T_pM) : \alpha \in A\Big\} \quad \text{with } y_\alpha \text{ as in (9).}$$

Moreover, this differentiable structure is independent of the atlas $\mathcal{A}$ chosen.

We call the manifold TM the *tangent bunde*.

Proof. Certainly $\mathcal{A}_{TM}$ covers TM. The topology on TM arises from taking open sets in $TM|U_\alpha := \bigcup_{p\in U_\alpha} T_pM$, which are preimages of open sets under y_α. arbitrary unions of such open sets form open sets in TM. With this topology, the charts y_α become homeomorphisms.

Let us now show that two charts are differentiably compatible. Let $p \in U_\alpha \cap U_\beta$ and $[c] \in T_pM$, where $p = c(0)$. Then

$$\begin{aligned}\Big(y_\beta \circ y_\alpha^{-1}\Big)\Big(x_\alpha(c(0)), \frac{d}{dt}(x_\alpha \circ c)\Big|_{t=0}\Big) &= y_\beta\big([c]\big) = \Big(x_\beta(c(0)), \frac{d}{dt}(x_\beta \circ c)|_{t=0}\Big) \\ &= \Big((x_\beta \circ x_\alpha^{-1} \circ x_\alpha)(c(0)),\ \frac{d}{dt}(x_\beta \circ x_\alpha^{-1} \circ x_\alpha \circ c)\Big|_{t=0}\Big) \\ &= \Big((x_\beta \circ x_\alpha^{-1})(x_\alpha(c(0))),\ d(x_\beta \circ x_\alpha^{-1})\frac{d}{dt}(x_\alpha \circ c)\Big|_{t=0}\Big)\end{aligned}$$

But $x_\beta \circ x_\alpha^{-1}$ is differentiable as a transition map, and the differential $d(x_\beta \circ x_\alpha^{-1})$ is smooth as the differential of a smooth map. □

Remark. The tangent bundle can be described as a special case of a *vector bundle*. Such a bundle locally is a product of a base space times a fixed vector space, in our case the product $U_\alpha \times \mathbb{R}^n$. However, globally the bundle need not be a product: For instance $T\mathbb{S}^2$ cannot be a product since each vector field on $\mathbb{S}^2$ has a zero.

3. Differentiable maps between manifolds

3.1. **The differential.** Remember that by definition a mapping between manifolds is differentiable if the composition with charts is a differentiable Euclidean map. We can now define its Jacobian:

Definition. Let $f\colon M \to N$ be a differentiable mapping between two manifolds. The *differential* (or tangent map) is the map

$$df\colon TM \to TN, \qquad df[c] := [f \circ c],$$

where c represents a tangent vector in $T_{c(0)}M$ and $f \circ c$ gives a tangent vector in $T_{f(c(0))}N$. We also write $df_p\colon T_pM \to T_{f(p)}N$ for the restriction of df.

Thus the class of c gets mapped to the class of $f \circ c$. Other notations for df involve f_* ("push-forward"), f', or Tf.

Let us show that df is well-defined. So suppose two curves c_1, c_2 through $p \in M$ satisfy $[c_1] = [c_2]$. Then, for $i = 1, 2$,

$$\frac{d}{dt}(y \circ f \circ c_i)(0) = \frac{d}{dt}(y \circ f \circ x^{-1}) \circ (x \circ c_i)(0) = d(y \circ f \circ x^{-1})_{x(p)} \cdot \frac{d}{dt}(x \circ c_i)(0), \tag{10}$$

and so indeed also $[f \circ c_1] = df[c_1]$ agrees with $[f \circ c_2] = df[c_2]$.

We can now assert properties which are well-known for the Euclidean case.

Theorem 5. *The differential $df\colon TM \to TN$ is a differentiable map. For each $p \in M$, the restrictions $df_p\colon T_pM \to T_{f(p)}N$ are linear.*

Proof. To prove differentiability we need to compose TM and TN with charts. In the resulting commutative diagramme we need to check that principal part of the image depends on the principal part of the preimage in a differentiable way. We leave this as an exercise.

To prove linearity, consider (10). The principal part of $[f \circ c]$ depends linearly on the principal part of $[c]$. But the map from tangent vectors to principal parts is an isomorphism, and so composing with these produces a linear map again. □

Theorem 6. *The chain rule $d(f \circ g)|_p = df_{g(p)} \circ dg_p$ holds for differentiable maps between manifolds.*

This is immediate from $\big[(f \circ g) \circ c\big] = \big[f \circ (g \circ c)\big] = df[g \circ c] = df\big(dg[c]\big) = (df \circ dg)[c]$, at the appropriate points.

`4. Lecture, Thursday 5.11.09` ______

Theorem 7. *Let $f\colon M^m \to N^n$ be a differentiable map. If df_p is a vector space isomorphism then $m = n$, and p has a neighbourhood W, such that $f|_W$ is a diffeomorphism onto its image.*

Proof. Pick charts x at p and y at $f(p)$. Then the inverse mapping theorem *[Umkehrsatz]* proves that $x(p)$ has a neighbourhood $W'\big(x(p)\big) \subset \mathbb{R}^n$ with $y \circ f \circ x^{-1}$ diffeomorphism. Since charts are diffeomorphisms, $W := x^{-1}(W')$ satisfies the claim. □

A *local diffeomorphism* is a map $f\colon M \to N$ for which the statement holds at each point, that is, each $p \in M$ has a neighbourhood W so that $f|_W\colon W \to f(W)$ is a diffeomorphism.

Example. $t \mapsto e^{it}$ is a local, but not a global diffeomorphism between $\mathbb{R}$ and $\mathbb{S}^1 \subset \mathbb{C}$.

Remark. To see that homeomorphisms of topological manifolds preserve dimension is much harder. First, this poses the problem to define a topological dimension; see, for instance § 50 of Munkres. But even in Euclidean spaces it requires tools from algebraic topology to prove that dimension is preserved by homeomorphisms. Only in dimenions 1, this is easy to see. Consider a homeomorphism f from an open interval I to an open connected set $U \subset \mathbb{R}^n$ where $n \geq 2$. For any $p \in I$, the set $I \setminus \{p\}$ is not connected. But for a homeomorphism, $f(I \setminus \{p\}) = f(I) \setminus \{f(p)\} = U \setminus \{f(p)\}$ is still connected, contradiction.

3.2. **Immersions and embeddings.**

Definition. Let $f\colon M \to N$ be a differentiable map between manifolds M and N.
(i) f is an *immersion* if its differential $df_p\colon T_pM \to T_{f(p)}N$ is injective for all $p \in M$.
(ii) f is an *embedding [Einbettung]*, if $f\colon M \to N$ is an immersion and a homeomorphism onto its image.

For (ii), the topology of the subset $Y := f(M) \subset N$ is the *subspace topology*: If $\mathcal{O}_N$ are the open sets of N then $\mathcal{O}_Y := \{U \cap Y : U \in \mathcal{O}\}$. It is easy to see that this is a topology.

In fact, in Euclidean space a differentiable homeomorphism with invertible differential is a diffeomorphism. Hence, in the situation (ii) we can also conclude that f is a diffeomorphism onto its image.

Examples (curves in $N = \mathbb{R}^2$): 1. A curve $c\colon I \to M$ is an immersion provided $dc_t = c'(t) \neq 0$. The differentiable curves $\mathbb{R}^2$, $t \mapsto (t^2, t^3)$ and $t \mapsto (t^3, t^3)$ are not immersions.

2. A *figure eight [Lemniskate]* can be parameterized by the immersion $c\colon \mathbb{S}^1 \to \mathbb{R}^2$, $c(e^{it}) = (\sin t, \sin 2t)$. This curve is not injective, hence not an embedding.

3. $e^{it}\colon \mathbb{R} \to \mathbb{C}$ is an immersion but not an embedding.

4. Consider an injective curve $c\colon (0,1) \to \mathbb{R}^2$ with a point of contact, e.g. $\lim_{t\to 0} c(t) = c(\frac{1}{2})$. Then preimages of sufficiently small neighbourhoods of $c(\frac{1}{2})$ consist of two connected components: A neighbourhood of 0 and one of $\frac{1}{2}$. But homeomorphisms preserve the number of components. Therefore, c is not a homeomorphism onto its image, that is, not an embedding.

5. A line with irrational slope in $\mathbb{R}^2$ projects to the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ as an injective immersion which is not an embedding. (The precise definition of the torus will be given only later.)

The reason to demand a homeomorphism in the definition of an embedding, not just an injective map, is that it is desirable to have the topology on the image to agree with the topology on the preimage. This property is violated by Examples 4 and 5.

If a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^l$ is injective, then $n \le l$. Thus immersions $f\colon M^n \to N^l$ have *codimension* $k = l - n \ge 0$. Locally, an immersion is an embedding:

Theorem 8. *Let $f\colon M^n \to N^{n+k}$ be an immersion. Then each $p \in M$ has a neighbourhood $W \subset M$ such that $f|_W$ is an embedding.*

Proof. We will invoke the inverse mapping theorem. We choose charts (x, U) at p and (y, V) at $f(p)$. Upon shrinking U if necessary we can assume $f(U) \subset V$. The local representation of f then reads

$$\varphi := y \circ f \circ x^{-1}\colon x(U) \to y(V), \qquad \varphi(u) = \Big(\varphi_1(u_1,\dots,u_n),\ \dots,\ \varphi_{n+k}(u_1,\dots,u_n)\Big).$$

We assume $x(p) = 0 \in \mathbb{R}^n$. We have $\operatorname{rank}(d\varphi) = n$ and so the $(n+k)\times n$-Jacobian $\big(\frac{\partial\varphi_i}{\partial u_j}(0)\big)$ has an $n\times n$-minor with rank n. Renumbering our φ-coordinates we may assume that the $n\times n$-matrix $\big(\frac{\partial\varphi_i}{\partial u_j}(0)\big)_{1\le i,j\le n}$ has rank n. Let us then set $\psi\colon x(U)\times\mathbb{R}^k \to \mathbb{R}^{n+k}$,

$$\psi(u_1,\dots,u_n,t_1,\dots,t_k) := \big(\varphi_1(u),\dots,\varphi_n(u),\varphi_{n+1}(u)+t_1,\dots,\varphi_{n+k}(u)+t_k\big), \tag{11}$$

then $\psi(u,0) = \varphi(u)$, and the Jacobian

$$J\psi = \begin{pmatrix} \left(\frac{\partial\varphi_i}{\partial u_j}\right)_{1\le i,j\le n} & 0 \\ \left(\frac{\partial\varphi_{n+i}}{\partial u_j}\right)_{1\le i\le k,\ 1\le j\le n} & 1_k \end{pmatrix}$$

has rank n at the point $(u,t) = (0,0)$, due to determinant development.

By the inverse mapping theorem there exists a neighbourhood $\Omega \subset x(U)\times\mathbb{R}^k$ of $0 \in \mathbb{R}^{n+k}$, such that ψ maps Ω diffeomorphically to $\psi(\Omega) \subset \mathbb{R}^{n+k}$. Let $\Omega \cap (x(U)\times\{0\}) \supset W'\times\{0\}$. Then the restriction $\psi|_{W'\times\{0\}} = \varphi|_{W'}$ is a homeomorphism of W' onto its image in $\mathbb{R}^{n+k}$. But charts are homeomorphisms and so the restriction of $f = y^{-1}\circ\varphi\circ x$ to $W := x^{-1}(W')$ is a homeomorphism onto its image, hence an embedding. $\square$

Remarks. 1. Since a bijective continuous mapping of a compact space to a topological manifold is a homeomorphism we have: If M is compact and $f\colon M \to N$ an injective immersion, then f is an embedding.

2. For later reference, let us state the following consequence of the proof:

Lemma 9. *If $f\colon M^n \to N^{n+k}$ is an immersion then for each $p \in M$ there exists a chart (x, U) of M at p and a chart $(\tilde{y}, V)$ of N at $f(p)$, such that*

$$\big(\tilde{y} \circ f \circ x^{-1}\big)(u_1, \ldots, u_n) = (u_1, \ldots, u_n, 0, \ldots, 0) \in \mathbb{R}^{n+k}.$$

Proof. Note that y and ψ are local diffeomorphisms and set

$$\tilde{y} := \psi^{-1} \circ y \,:\; y^{-1}(\psi(\Omega)) \to \Omega.$$

With respect to the chart $\tilde{y}$ we have the following local representation of f:

$$\tilde{y} \circ f \circ x^{-1} = (\psi^{-1} \circ y) \circ f \circ x^{-1} = \psi^{-1} \circ \varphi.$$

Since $\psi(u, 0) = \varphi(u)$ we have $(\psi^{-1} \circ \varphi)(u) = (u, 0)$, as claimed. □

3.3. **Submanifolds.** There are various ways to define n-dimensional submanifolds of Euclidean space $\mathbb{R}^{n+k}$ locally:
(i) As the inverse image of a regular value of a function to $\mathbb{R}^k$,
(ii) the image of the slice $\mathbb{R}^n \times \{0\}$ where all of $\mathbb{R}^n \times \mathbb{R}^k$ parameterizes diffeomorphically an open set in ambient space,
(iii) being parameterized with $\mathbb{R}^n$.
These characterization can also be given for submanifolds contained in ambient manifold, and are again equivalent. Here, we turn the second characterization into a definition:

Definition. A connected subset $M^n \subset N^{n+k}$, $k \geq 0$, is an n-dimensional *submanifold [Untermannigfaltigkeit]* of N if at each $p \in M$ there is a chart $y\colon U \to \mathbb{R}^{n+k}$ of N subject to to

$$y(M \cap U) = y(U) \cap \big(\mathbb{R}^n \times \{0\}\big). \tag{12}$$

For V open, we call a set $V \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^k$ a *slice.* Then (12) says that the charts y map the submanifold M locally to a slice. A submanifold keeps a distance from itself: All points of M in the set U are mapped to the slice.

A submanifold $M^n \subset N^{n+k}$ is a manifold in its own right. To see this, let (y_α, U_α) be an atlas of charts of N subject to (12). Then the charts $(x_\alpha, U_\alpha \cap M)$ with

$$x_\alpha\colon U_\alpha \cap M \to \mathbb{R}^n, \qquad x_\alpha(p) = \big(y^1_\alpha(p), \ldots, y^n_\alpha(p)\big)$$

certainly cover M. Then $x_\beta \circ x_\alpha^{-1} = y_\beta \circ y_\alpha^{-1}$ is differentiable as a coordinate restriction of a differentiable map. Moreover, the set M inherits the Hausdorff and second countability property from N.

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We will need the following result:

Theorem 10. *If $f\colon M \to N$ is an embedding then its image $f(M) \subset N$ (with the subspace topology) is a submanifold of N.*

Proof. We need to show that each point of $f(M)$ has a neighbourhood W such that a chart (y, W) maps $f(M) \cap W$ to the slice $f(W) \cap (\mathbb{R}^n \times \{0\})$.

As a result of Lemma 9, charts (x, U) at p and $(\tilde{y}, \tilde{V})$ at $f(p)$ represent $\tilde{y}\big(f(U)\big)$ as an n-dimensional slice in $\mathbb{R}^{n+k}$.

But f is a homeomorphism of M onto its image $f(M)$. Hence the open set $U \subset M$ has an open image $f(U)$ in $f(N)$. By definition of the subspace topology this means there is an open set $V \subset N$ such that $V \cap f(M) = f(U)$. But then $W := V \cap \tilde{V}$ is an open set, such that the restriction y of $\tilde{y}$ maps $f(M) \cap W = f(U) \cap W$ to a slice. $\square$

4. The Whitney embedding theorem

Whitney's theorem from 1944 says that any differentiable n-manifold can be embedded into $\mathbb{R}^{2n}$. Hence the class of abstract manifolds is no larger than the class of submanifolds of Euclidean space! Nevertheless, for constructions such as quotients, it is much more natural to work with abstract manifolds than with an immersion – for instance, it takes some work to figure out the explicit form of an embedding of $\mathbb{R}P^2$ into $\mathbb{R}^4$.

We will provide the proof for a slightly weaker result, an immersion into $\mathbb{R}^{2n}$ and an embedding into $\mathbb{R}^{2n+1}$, which was what Whitney had proved first, in 1936. We will also concentrate on the case of compact manifolds. Then a finite number of charts suffices, which simplifies the proof. See [Lee], Chapter 10, for the details of the general, non-compact case.

As the example of a curve with a double point indicates, it is easy to remove a double point in $\mathbb{R}^{2n+1}$ since there is room enough to move one n-dimensional branch of the manifold off another n-dimensional branch. Supose we already have an immersion $f\colon M \to \mathbb{R}^{2n+1}$. Since it is locally an embedding we can assume that on each chart of M the map f is an embedding. To remove the self-intersection points of different charts, we add small constants to each chart, and mollify the result by a partition of unity. For almost all constants this works, as the previous intuition tells us.

It is surprising that a version of the same idea also lets us perturb a given map from M into $\mathbb{R}^{2n}$ (perhaps a constant) into an immersion. For this, we perturb the given map on each chart by adding a map which is linear on the coordinates of the chart and using a partition of unity. In local coordinates, the Jacobian is given by the constant matrix A plus the Jacobian of the given map. For points ranging in the chart, the image $\{df_p + A : p \in U\}$ is an n-dimensional set in the space of $2n \times n$-matrices. We can choose A such that this n-dimensional set is disjoint from the space of matrices with rank $r = n-1$ or lower. To see that we will show that the space of these singular matrices has codimension $n+1$ in the space of all matrices. Hence there is a dense subset of matrices, such that the perturbed maps miss the set of matrices with rank $r = n-1$ or lower.

I used Lee's book to prepare this section.

4.1. **Matrices of fixed rank.** Let $\mathsf{M}(m \times n)$ denote the space of real m by n matrices with real coefficients; as usual we will consider the matrices as a subset of Euclidean space, $\mathsf{M}(m \times n, \mathbb{R}) = \mathbb{R}^{mn}$.

Proposition 11. *For each $0 \le r \le \min\{m,n\}$ the space of $m \times n$-matrices $\mathsf{M}_r(m \times n)$ with rank equal to r is the union of submanifolds of the space $\mathsf{M}(m \times n) = \mathbb{R}^{mn}$ with*

$$\dim \mathsf{M}_r(m \times n) = r(n+m-r). \tag{13}$$

Note that for the case of maximal rank, $r = \min\{m,n\}$, the dimension is mn, as expected. Moreover, the dimension formula is symmetric in m, n.

Proof. Let us first consider the set

$$U := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{M}(m \times n) : A \in \mathsf{M}(r \times r) \text{ satisfies } \det A \neq 0 \right\}.$$

The determinant function is continuous and so the set U is an open subset of $\mathsf{M}(m \times n)$.

Let us now give a condition for a matrix $M \in U$ to have rank exactly equal to r. We transform M into a standard echelon form *[Stufenform]* by multiplying it from the right with a suitable invertible $n \times n$-matrix in block matrix form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A^{-1} & -A^{-1}B \\ 0 & 1_{n-r} \end{pmatrix} = \begin{pmatrix} 1_r & 0 \\ CA^{-1} & D - CA^{-1}B \end{pmatrix} \in U$$

Clearly,

$$\operatorname{rank} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = r \quad \Leftrightarrow \quad D - CA^{-1}B = 0.$$

Hence if we define the smooth function

$$\Phi\colon U \to \mathsf{M}\big((m-r)\times(n-r)\big), \qquad \Phi\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) := D - CA^{-1}B$$

then $\mathsf{M}_r(m\times n)\cap U = \Phi^{-1}(0)$.

To show that $\mathsf{M}_r(m\times n)\cap U$ is a submanifold of $\mathbb{R}^{nm}$ we claim that the 0-matrix is a regular value of Φ. This means to show that $d\Phi$ is surjective. To do so, let $S\in\mathsf{M}\big((m-r)\times(n-r)\big)$ be arbitrary. Then the curve

$$M(t) := \begin{pmatrix} A & B \\ C & D+tS \end{pmatrix} \in U \quad \text{satisfies} \quad \Phi\big(M(t)\big) = (D-CA^{-1}B)+tS,$$

and so the linearization of Φ at $M(0)$ is

$$d\Phi_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)}\left(\begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}\right) = S.$$

This proves that $d\Phi$ is surjective, as desired.

Now pick an arbitrary $M\in\mathsf{M}_r$. Then some $(r\times r)$-minor of M has rank r. Since the determinant of the minor is nonzero, this same minor has rank r for all matrices in M_r in some neighbourhood V of M.

Reindexing coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$, so that the minor with rank r of M maps to the top left $r\times r$ minor gives a map $\Psi\colon V\subset\mathbb{R}^{mn}\to U\subset\mathbb{R}^{mn}$. Clearly, Ψ is a diffeomorphism onto its image. The above reasoning shows that $V=(\Phi\circ\Psi)^{-1}(0)$ is a submanifold of $\mathsf{M}(m\times n)=\mathbb{R}^{mn}$ in the neighbourhood V of M. That is, we have determined a chart (y,V) at M. This proves that M_r is a submanifold altogether.

Certainly $\dim\mathsf{M}_r = \dim\mathsf{M}(m\times n) - \dim\mathsf{M}((m-r)\times(n-r)) = mn-(mn-(m+n-r)r) = (m+n-r)r$. Moreover, it can be shown that M_r is connected unless $r=m=n$ in which case there are two components (see problems). □

We will later see that the essential condition for a given manifold M^n to embed into some space $\mathbb{R}^{m\geq n}$ is that the space of the singular $(m\times n)$-matrices, that is, of matrices with rank $r<n$, has codimension at least $n+1$ in the space of all matrices. This is precisely the case for $m\geq 2n$:

Corollary 12. *For $r\leq n\leq m$ let $\operatorname{codim}\mathsf{M}_r(m\times n) := \dim\mathsf{M}(m\times n) - \dim\mathsf{M}_r(m\times n)$. Then*

$$\operatorname{codim}\mathsf{M}_r(m\times n)\geq n+1 \quad \textit{for all } \ 0\leq r\leq n-1 \quad \textit{if and only if} \quad m\geq 2n.$$

Proof. First we claim that $0 = \dim \mathsf{M}_0 \leq \ldots \leq \dim \mathsf{M}_n$. Indeed, the dimension (13) increases in r as

$$r \leq n \leq m \quad \Rightarrow \quad \frac{d}{dr} r(n+m-r) = n+m-2r \geq 0.$$

Consequently, using Prop. 11 with $r = n-1$,

$$\begin{aligned}\operatorname{codim} \mathsf{M}_0(m \times n) \geq \cdots \geq \operatorname{codim} \mathsf{M}_{n-1}(m \times n) &\overset{(13)}{=} mn - (n-1)(n+m-n+1) \\ &= mn - (n-1)(m+1) = m-n+1 \overset{!}{\geq} n+1\end{aligned}$$

if and only if $m \geq 2n$. □

4.2. **Sets of measure zero.** We consider the Lebesgue measure λ on Euclidean space. A set $A \subset U$ for U open in $\mathbb{R}^n$ has measure 0 *[Nullmenge]* if for each $\varepsilon > 0$ there are countable many measurable sets S_i which cover $A \subset \bigcup_{n \in \mathbb{N}} S_i$ and have total measure $\sum_{i \in \mathbb{N}} \lambda(S_i) < \varepsilon$. For instance, the coordinate subspaces $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+k}$ have measure 0 whenever $k \geq 1$.

Lemma 13. *Let $f \colon U \to V$ be a differentiable map between open subsets of Euclidean spaces.*
(i) Suppose $U, V \subset \mathbb{R}^n$ and $A \subset U$ has measure zero. Then $f(A)$ has measure zero.
(ii) Suppose $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^{n+k}$ with $k > 0$. Then $f(U)$ has measure zero.

Proof. (i) Let the sets S_i cover A with total measure less than ε. We first consider the case that all S_i are contained in a compact set $K \subset U$. Then the map f has bounded differential $\|df_x\| \leq C = C(K)$ for all $x \in K$ and so is Lipshitz. Then $\lambda(f(S_i)) \leq C^n \lambda(S_i)$, and so also $\lambda\big(f(A)\big) \leq C^n \lambda(A)$ which proves the statement for this case.

`6. Lecture, Thursday 19.11.09` ________________________________

Each open set U has an excision *[Ausschöpfung]* with compact sets

$$K_1 \subset \ldots \subset K_j \subset \ldots \subset \bigcup_{j=1}^{\infty} K_j = U,$$

for instance $K_j := \{x \in U : \|x\| \leq j \text{ and } B_{1/j}(x) \subset U\}$. Then $A_j := A \cap K_j$ has a countable covering with the sets $T_{ij} := S_i \cap K_j$ which are measurable and $\sum_{i=0}^{\infty} \lambda(T_{ij}) \leq \sum_{i=0}^{\infty} \lambda(S_i) \leq \varepsilon$. The sets $(T_{ij})_{i \in \mathbb{N}}$ are contained in the compact set K_j, and so by the argument in the first paragraph the set $f(A_j)$ has measure 0. But $f(A) = f\big(\bigcup_{j=1}^{\infty} A_j\big) = \bigcup_{j=1}^{\infty} f(A_j)$ is a countable union of sets of measure 0, and hence has measure 0 itself. (I thank Miroslav Vrzina for suggesting this argument.)

(ii) Extend f to a differentiable map $F \colon U \times \mathbb{R}^k \to \mathbb{R}^{n+k}$ by setting $F(x,y) := f(x)$. Then $A := U \times \{0\} \subset \mathbb{R}^n \times \{0\}$ is a set of measure 0 in $\mathbb{R}^{n+k}$, and so by (i) the image $F(A) = f(U)$ has measure 0 in $\mathbb{R}^{n+k}$. □

Note, however, that our lemma fails to hold for continuous mappings. For instance, a space filling curve $c(t)$ maps the unit interval $[0,1]$ onto the square $[0,1] \times [0,1]$ with larger dimension. Nevertheless the image has measure one, in contrast to statement (ii). (Construct a continuous counterexample to (i)!)

Statement (i) of the lemma says that having measure zero is a property invariant under diffeomorphisms. This makes sets of measure zero well-defined on manifolds, that is, independent of the choice of atlas:

Definition. A subset $A \subset M^n$ has measure 0 if for all charts (x_α, U_α) of an atlas $\mathcal{A}$ the sets $x_\alpha(A \cap U_\alpha)$ have measure 0 in $\mathbb{R}^n$.

Indeed, if (y_β, V_β) is another atlas, then using the axiom of second countability, $A \cap V_\beta$ is covered by a countable union of charts (x_α, U_α) and hence $y_\beta(A \cap V_\beta)$ has measure 0.

We can now formulate case (ii) of the lemma for manifolds:

Theorem 14. *For $k > 0$ let $f \colon M^n \to N^{n+k}$ be a smooth map of a manifold M into a manifold N. Then $f(M)$ has measure 0 in N and so has a dense complement $N \setminus f(M)$.*

Proof. Restricted to the appropriate charts, we get a differentiable Euclidean map from dimension n to dimension $n+k$ of the form

$$y_\beta \circ f \circ x_\alpha^{-1} \colon \quad x_\alpha\bigl(f^{-1}(V_\beta) \cap U_\alpha\bigr) \to y(V_\beta).$$

Thus by Lemma 13(ii) the image is a set of measure 0. But by definition this means that $f(M)$ has measure 0 in N. $\square$

Examples. 1. The space M_r of matrices with rank $0 \leq r < \min\{m,n\}$ forms a set of measure 0 in the space $\mathsf{M}(m \times n) = \mathbb{R}^{mn}$.

2. A straight line in the torus T^2 is the image of an immersion $f \colon \mathbb{R}^1 \to T^2$. Hence the theorem says its complement $T^2 \setminus f(\mathbb{R})$ is dense in the torus. Note that this reasoning is true for irrational slope in which case the image is not a submanifold of T^2. In that case, the situation is very similar to the rational numbers as a subset of $\mathbb{R}$ with dense complement.

4.3. **The immersion theorem.** We want to perturb a given map f to an immersion h. To display the essential idea of the proof, we start with a simple case:

Lemma 15. *Let M^n be a manifold with one chart and let $f \colon M^n \to \mathbb{R}^{2n}$ be a smooth map. Then, for any $\varepsilon > 0$, there is a smooth immersion*

$$h \colon M^n \to \mathbb{R}^{2n} \quad \textit{such that} \quad \sup_M |f - h| < \varepsilon. \tag{14}$$

Proof. We assume the chart (x, M) maps into the unit ball of $\mathbb{R}^n$. We will change f by a function which is linear in the coordinates of the chart,

$$h(p) := f(p) + \varepsilon A x(p), \qquad \text{with} \qquad A \in \mathsf{M}(2n \times n),\ \|A\| \le 1,$$

where A is yet to be determined. Since $\|x(p)\| < 1$ for all $p \in M$, the condition $\|A\| \le 1$ ensures that (14) holds.

We want to determine A such that h is an immersion. By definition, this means that the map $h \circ x^{-1} \colon x(M) \to \mathbb{R}^{2n}$ is an immersion. To write everything on $x(M)$, we replace p by $x^{-1}(u)$ and set

$$\tilde{h}(u) := (h \circ x^{-1})(u) = (f \circ x^{-1})(u) + \varepsilon A x(x^{-1}(u)) =: \tilde{f}(u) + \varepsilon A u.$$

We need to prove:

$$\exists A: \quad d\tilde{h}_u = d\tilde{f}_u + \varepsilon A \qquad \text{has rank } n \text{ for each point } u \in x(M). \tag{15}$$

Let us give the idea. The map $d\tilde{f}_u$ has an n-dimensional image in $\mathsf{M}(2n \times n)$. On the other hand, the space of matrices $\mathsf{M}_r(2n \times n)$ with rank $r < n$ has codimension at least $n+1$. (This is only true for target dimension $2n$ or larger). So if for some choice of A the matrix-valued map $u \mapsto d\tilde{h}_u = d\tilde{f}_u + \varepsilon A$ hits the submanifolds $\mathsf{M}_r(2n \times n)$ of singular matrices then a slight perturbation of A will move the image away from $\mathsf{M}_r(2n \times n)$. To make this argument precise, let us drop the matrix dimensions $2n \times n$ from now on.

Our task is to pick $A = \frac{1}{\varepsilon}(d\tilde{h}_u - d\tilde{f}_u)$ such that $d\tilde{h}_u$ is not a matrix of rank $n-1$ or lower. That is, for

$$Q \colon x(M) \times \mathsf{M} \to \mathsf{M}, \qquad Q(u, B) := \frac{1}{\varepsilon}(B - d\tilde{f}_u)$$

we desire that

$$A \ne Q(u, B) \qquad \text{for all} \quad u \in x(M) \quad \text{and for all} \quad B \in \mathsf{M}_0 \cup \ldots \cup \mathsf{M}_{n-1}.$$

Equivalently, A is not an element of

$$Q\Big(x(M) \times \big(\mathsf{M}_0 \cup \ldots \cup \mathsf{M}_{n-1}\big)\Big) = Q\big(x(M) \times \mathsf{M}_0\big) \cup \ldots \cup Q\big(x(M) \times \mathsf{M}_{n-1}\big). \tag{16}$$

By Corollary 12 the product of Euclidean submanifolds $x(M) \times \mathsf{M}_r$ has dimension at most $n + \big(\dim(\mathsf{M}_n) - (n+1)\big) = \dim \mathsf{M}_n - 1$ whenever $r \le n-1$. Applying Thm. 14 with $k = k(r) \ge 1$ we see that the image of any of these product manifolds in the $2n^2$-dimensional manifold $\mathsf{M} = \mathbb{R}^{2n^2}$ is a set of measure zero. Thus the complement of (16) is dense and so there exists A in this complement with norm less than 1 (or even less than any given positive norm). For any such A then (15) holds and so h is an immersion. $\square$

The statement of Lemma 15 holds for arbitrary manifolds. For reasons of simplicity, let us discuss here the compact case only. Note that for a compact manifold, an open covering always has a finite subcover. Moreover, let us state without proof the following fact: If a compact manifold M is covered with open sets V_k, then it is also covered with open sets U_k, such that $\overline{U}_k \subset V_k$ is compact.

Proposition 16. *The conclusion of Lemma 15 holds for M compact.*

In particular, choosing $f \equiv 0$, we obtain an immersion $h\colon M \to \mathbb{R}^{2n}$ of a given compact manifold M.

Proof. Let $\{(x_k, V_k)\colon 1 \le k \le \ell\}$ be a finite atlas of our compact manifold M. Again we may assume that $x_k(V_k) \subset B_1 \subset \mathbb{R}^n$. We pick sets $\overline{U}_k \subset V_k$ which still cover M, and consider bump functions $0 \le \varphi_k \le 1$, which are 1 on U_k and have $\operatorname{supp} \varphi_k \subset V_k$. Note that $\sum \varphi_k/\ell \le 1$.

We construct functions $h_0 := f, h_1, \ldots, h_\ell =: h$, by perturbing f in each chart by a function linear in the coordinates, and piece the result together using our bump functions: So for $k = 0, 1, \ldots, \ell$, let

$$h_k(p) := f(p) + \frac{\varepsilon}{\ell} \sum_{i=1}^{k} \varphi_i(p) A_i x_i(p), \qquad \text{with} \qquad A_i \in \mathsf{M}(2n \times n),\ \|A_i\| \le 1,$$

where A_i is yet to be determined. Here we assume that each term of the sum has been extended with value 0 to all of M. Since $\|x(p)\| < 1$ for all $p \in M$, the condition $\|A_i\| \le 1$ will ensure that (14) holds.

We now show iteratively for $k = 1, \ldots, \ell$:

$$\exists A_k : \quad h_k = h_{k-1} + \frac{\varepsilon}{\ell} \varphi_k A_k x_k \quad \text{is an immersion on } \overline{U}_1 \cup \ldots \cup \overline{U}_k. \tag{17}$$

Note that for $k = \ell$ this says that $h := h_\ell$ is an immersion on all of M, as desired. So we suppose that (17) holds for $k-1$ and establish it for k.

• For the subset $\overline{U}_1 \cup \ldots \cup \overline{U}_{k-1}$: Note that $dh_k = dh_{k-1} + \frac{\varepsilon}{\ell} A_k$, where dh_{k-1} has rank n. By continuity of the determinant function on minors and compactness of our set we can find $\delta_k \in (0,1)$, such that for any A_k with $\|A_k\| < \delta_k$ the map dh_k still has rank n.

• For the subset $\overline{U}_k$: The dimension count of the proof of Lemma 15 shows that we can achieve $h_k = h_{k-1} + \frac{\varepsilon}{\ell} A_k x_k$ to have rank n on $\overline{U}_k$ for some matrix A_k, even under our constraint $\|A_k\| \le \delta_k$. □

Remark. Using more subtle arguments it can be shown that each n-manifold for $n \ge 2$ can actually be immersed into $\mathbb{R}^{2n-1}$ (Whitney 1944).

`7. Lecture, Thursday 26.11.09` ______________________________

4.4. **Some topology.** Our goal is the assertion:

Proposition 17. *For M a compact manifold, any injective immersion $f\colon M \to N$ is an embedding.*

The closedness of f is the key to proving this property. A map $f\colon X \to Y$ between topological spaces is a *closed map* if each closed subset $A \subset X$ has a closed image $f(A) \subset Y$.

Example. Recall Example 4 from p. 13: This immersion f of the open interval into $\mathbb{R}^2$ (with "touching point") is not closed: The (relatively) closed set $[1-\varepsilon, 1)$ is mapped to a set in $\mathbb{R}^2$ which is not closed since it fails to contain the limit of the sequence $f(1-1/n)$.

Lemma 18. *If $f\colon X \to Y$ is a continuous map of topological spaces, where X is compact and Y is Hausdorff, then f is closed.*

Proof. Our claim follows from three topological facts, whose proof we leave as an exercise:
- A closed subset A of a compact space X is compact itself.
- A compact set A has a continuous image $B := f(A)$ which is also compact.
- A compact subset B of a Hausdorff space Y is closed. $\square$

Proof of the Proposition. We need to show that f is a homeomorphism; in fact, we need to show f^{-1} is continuous.

We use two facts:
- By definition, closed sets are complements of open sets. For a subspace $X \subset Y$ with the subspace topology, a subset $A \subset X$ is closed if there is a closed set $B \subset Y$, such that $A = B \cap X$.
- A mapping $f\colon X \to Y$ is continuous if and only if closed sets in Y have preimages which are closed in X.

By injectivity, $f^{-1}\colon f(M) \to M$ exists. By the lemma, if $A \subset M$ is closed then $f(A)$ is closed in Y. By our first fact this means $f(A)$ is closed in $f(M)$ as well. Thus f^{-1} has the property that the preimages of closed sets are closed. By the second fact, f^{-1} is continuous. $\square$

4.5. **The embedding theorem.** To perturb an immersion of an n-dimensional manifold in $\mathbb{R}^{2n+1}$ to an embedding seems easy: Using the extra dimension we have, we can move one branch of the surface away from any other at self-intersections.

Again we specialize to the compact case:

Theorem 19. (i) *Let M be compact. Given an immersion $f\colon M \to \mathbb{R}^{2n+1}$ and $\varepsilon > 0$ there is an embedding $h\colon M \to \mathbb{R}^{2n+1}$, such that $|h - f| \le \varepsilon$.*
(ii) *Every compact n-manifold admits an embedding to $\mathbb{R}^{2n+1}$.*

Proof. (i) Locally, an immersion is an embedding, by Thm. 8. Thus M has a covering with sufficiently small charts (x_k, V_k) such that $f(V_k)$ is an embedding. Since M is compact, a finite number ℓ of these charts suffices to cover. Again we choose sets $\overline{U}_k \subset V_k$ which still cover, and let φ_k be bump functions with support in V_k and which are identical to 1 exactly on U_k.

We set $h_0 := f$ and determine $h_1, \dots, h_\ell =: h$ by setting for $k = 1, \dots, \ell$

$$h_k(p) := h_{k-1}(p) + \frac{\varepsilon}{\ell}\varphi_k(p)\, b_k \qquad \text{where } b_k \in \mathbb{R}^{2n+1} \text{ with } \|b_k\| \leq 1$$

where the b_k are chosen such that
1. h_k is an immersion on M,
2. h_k is injective on each chart V_i, $i = 1, \dots, \ell$
3. h_k is injective on $\overline{U}_1 \cup \ldots \cup \overline{U}_k$.

Note that 1. and 2. hold for $k = 0$ (while 3. is vacuous). Now we assume 1.-3. for $k-1$ and pick a suitable b_k for them to hold for k, as follows.

For 1., we assume dh_{k-1} has rank n on M. Thus, by continuity of the determinant and compactness of M we can find $\delta_k > 0$, such that for any b_k with $\|b_k\| < \delta_k$ the map dh_k is also an immersion. Then 1. is satisfied.

Let us now pick b_k (subject to $\|b_k\| < \delta_k$) in such a way that we can rule out the case

$$h_k(p) = h_k(q) \qquad \text{under the assumption } \varphi_k(p) \neq \varphi_k(q), \qquad p, q \in M. \tag{18}$$

Therefore, 2. and 3. hold for this case. Then $h_{k-1}(p) + \frac{\varepsilon}{\ell}\varphi_k(p)\, b_k = h_{k-1}(q) + \frac{\varepsilon}{\ell}\varphi_k(q)\, b_k$ or

$$b_k = -\frac{\ell}{\varepsilon}\,\frac{h_{k-1}(p) - h_{k-1}(q)}{\varphi_k(p) - \varphi_k(q)} =: B(p, q).$$

Here, B is defined on the set

$$U := \{(p, q) \in M \times M : \varphi_k(p) \neq \varphi_k(q)\}.$$

But the differentiable map B maps the $2n$-dimensional set U into $\mathbb{R}^{2n+1}$. By Thm. 14 the image $B(U) \subset \mathbb{R}^{2n+1}$ has measure 0 and a dense complement. Therefore, we may pick $b_k \notin B(U)$ with $\|b_k\| < \delta_k$ and then (18) will not hold for any pair $p, q \in M$.

We are now in a position to prove 2. and 3. for the remaining case

$$h_k(p) = h_k(q) \quad \text{under the assumption } \varphi_k(p) = \varphi_k(q) \quad \text{implies} \quad h_{k-1}(p) = h_{k-1}(q).$$

This together with (18) shows that the inductive assumption 2. for h_{k-1} implies 2. for h_k.

Let us now prove 3. By our previous reasoning, it suffices to show

$$h_{k-1}(p) = h_{k-1}(q) \qquad \text{for } p, q \in \overline{U}_1 \cup \ldots \cup \overline{U}_k.$$

To rule this out for $p \neq q$, let us distinguish three cases:
- Suppose both points are in $\overline{U}_k$. Since $\overline{U}_k \subset V_k$ this is impossible by hypothesis 2.
- One point is in $\overline{U}_k$, the other is not. For instance, $p \in \overline{U}_k$ but $q \notin \overline{U}_k$. Then $\varphi_k(p) = 1$

but $\varphi_k(q) < 1$ which is also impossible.
• Otherwise $p, q \in \overline{U}_1 \cup \ldots \cup \overline{U}_{k-1}$ which contradicts the inductive hypothesis 3.

(ii) Choose $f \equiv 0$ in Prop. 16 and insert the resulting immersion in (i). □

Let us state a consequence of the embedding theorem. The restriction of the distance function $d(x, y) = |x - y|$ in $\mathbb{R}^{2n+1}$ to M preserves the properties of a metric, and hence gives:

Corollary 20. *Each compact manifold M carries a metric d such that (M, d) is a metric space.*

4.6. **Problems.**

Problem 1 – DIFFERENTIABLE STRUCTURES ON $\mathbb{R}$:

Consider the two differentiable atlases, each consisting of just one chart,

$$\mathcal{A} := \{(\mathrm{id}, M)\}, \qquad \mathcal{B} := \{(x^3, N)\}$$

of the topological manifold $M := N := \mathbb{R}$.

a) Verify that x^3 is indeed a chart for N.
b) Show that the differentiable structures on $\mathbb{R}$ which are determined by $\mathcal{A}$ and $\mathcal{B}$ are different.
c) Which of the two following maps from M to N are diffeomorphisms?
   • $f(x) = \sqrt[3]{x}$ • identity.

Note: There are pairs of differentiable structures that do not arise as a diffeomorphic image of oneanother (see c), for instance on $\mathbb{R}^4$ and many spheres. The non-standard structure is called an *exotic differentiable structure*.

Problem 2 – TWO DIFFERENTIABLE STRUCTURES ON $\mathbb{R}^2$:

Let $M := D = \{q \in \mathbb{R}^2 : \|q\|^2 < 1\} \subset \mathbb{R}^2$ be the open disk. We consider two charts of M: On the one hand, let $x\colon D \to D$ be the identity. On the other hand, we define a mapping from the disk to the square with edgelength 2, namely

$$y\colon D \to Q := \{q \in \mathbb{R}^2 \,:\, -1 < q_1, q_2 < 1\} \subset \mathbb{R}^2, \qquad y(q) = \begin{cases} r(q)q, & q \neq 0, \\ 0, & q = 0. \end{cases}$$

Here, $r\colon D \setminus \{0\} \to [1, \sqrt{2}]$ is chosen such that y maps the unit circle ∂D to the boundary of the square ∂Q.

$$\begin{tikzcd} & M=D \arrow[dl, "x = \mathrm{id}"'] \arrow[dr, "y"] & \\ D \arrow[rr, "y \circ x^{-1} = y"'] & & Q \end{tikzcd}$$

a) Verify that y is a chart for M.

b) Why do x and y each determine a differentiable structure on M?

c) Prove that the two charts x and y are not differentiably compatible. Therefore, the two differentiable structures are not compatible.

Problem 3 – FOLIATIONS AS NON-HAUSDORFF-SPACES:

A *foliation [Blätterung]* of $\mathbb{R}^n$ is a decomposition of the entire space $\mathbb{R}^n$ into disjoint submanifolds, called *leaves*, which all have the same dimension $0 < k < n$. See [Lee], p. 510, for a precise definition, and p. 511 for pictures of foliations which give intuition.

a) Set $x \sim y :\Leftrightarrow x$ and y are contained in the same leaf. Check that this defines an equivalence relation.

b) We let the *leaf space* be $\mathcal{F} = \mathbb{R}^n/\sim$ and define a topology on $\mathcal{F}$: A set $U \subset \mathcal{F}$ is open if the union of the leaves represented by U is open in $\mathbb{R}^n$. Convince yourself that this defines a topology.

c) From now on we consider the case $n = 2$ and $k = 1$. Consider the foliation of $\mathbb{R}^2$ by parallel lines. Show that $\mathcal{F}$ is homeomorphic to $\mathbb{R}$.

d) Foliate two disjoint half-spaces with parallel lines, and the strip inbetween with U-shaped curves (Reeb foliation). Show that $\mathcal{F}$ is non-Hausdorff.
Hint: Let $\ell_{1,2}$ be the two special lines bounding the strip. Represent the restriction of the foliation in the two halfspaces by rays, and the strip by an interval; reason for that! Now study neighbourhoods of the points representing $\ell_{1,2}$.

e) Increase now the number of Reeb components – what does $\mathcal{F}$ look like?

f) Can Reeb components be nested? *Hint*: The space in between two Reeb leaves is homeomorphic to an open strip.

g) Speculation: Convince yourself that leaves are never homeomorphic to $\mathbb{S}^1$, but always to $\mathbb{R}$, and that they leave each compact set. Guess how we could define a *non-Hausdorff tree* (are continuous curves defined in $\mathcal{F}$?) and give evidence that $\mathcal{F}$ has a structure of a non-Hausdorff tree.

Problem 4 – Minimal atlas:

a) Let M be a compact manifold, containing at least two points. Show that each atlas of M contains at least two charts.

b) In particular the stereographic atlas of $\mathbb{S}^n$ is minimal.

c) Find an atlas of the 2-torus consisting of two charts. Does the n-torus also have an atlas consisting of two charts?

Problem 5 – Tangent vectors to $\mathbb{S}^2$:

The following curves in $\mathbb{S}^2$ are defined in a neighbourhood of $t = 0$. Which curves are equivalent in $\mathbb{S}^2$ and define the same tangent vector?

$$c_1(t) = \big(\cos t, 0, \sin t\big) \qquad c_2(t) = \big(\sin t, 0, \cos t\big),$$
$$c_3(t) = \big(\cos(2t), 0, \sin(2t)\big), \qquad c_4(t) = \big(\sqrt{1-t^2}, 0, t\big)$$

Check first that $c_i(0)$ agrees, and then for a chart x that $(x \circ c)'(0)$ agrees. A good choice of x is projection to a coordinate plane (verify that x a chart!).

Problem 6 – Tangent vectors to $\mathbb{R}P^2$:

Consider the point $p = [1, 1, 0] \in \mathbb{R}P^2$ and the charts x_1 and x_2 given in the lecture.

a) Find curves $c_1(t)$, $c_2(t)$ in $\mathbb{R}^2$ which represent the standard basis at p w.r.t. x_1.

b) Decide if c_1, c_2 also represent the standard basis w.r.t. x_2. To do so, consider the representing curves $d_i(t) := (x_2 \circ x_1^{-1})(c_i(t))$ in the image of x_2.

c) Which linear mapping maps $c_i'(0)$ to $d_i'(0)$?

Problem 7 – Immersions and embeddings:

a) The improved form of the Whitney embedding theorem says that each n-manifold can be immersed into $\mathbb{R}^{2n-1}$, and embedded into $\mathbb{R}^{2n}$. Discuss these statements for the case $n = 1$.

b) Can a Möbius strip be embedded into $\mathbb{R}^3$?

c) Find a 2-manifold which cannot be embedded into $\mathbb{R}^3$ (and reason for this fact).

Problem 8 – Manifolds as metric spaces:

a) Consider an immersion $f\colon M \to \mathbb{R}^{n+k}$ of a manifold M^n. For $p, q \in M$, set $d(p, q) := |f(p) - f(q)|$. Is d a metric?

b) As before, but for the case that f is an embedding. Prove d defines a metric on M.

Problem 9 – HELICOIDS IN $\mathbb{S}^3$:

Let $a \in \mathbb{R}$ be a parameter and consider the mapping

$$h = h_a \colon \mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4, \qquad (u,v) \mapsto \begin{pmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \cos(av) \\ \sin u \sin(av) \end{pmatrix}.$$

a) h_a is an immersion of $\mathbb{R}^2$ for $a \neq 0$.
Hint: Calculate the determinant of a 2×2 minor of the Jacobian $Jh = \left(\frac{\partial}{\partial u}h, \frac{\partial}{\partial v}h\right)$.

b) Show the two axes $v \mapsto a_1(v) = h(0,v)$ and $v \mapsto a_2(v) = h(\frac{\pi}{2}, v)$ are great circles whose points are pairwise perpendicular. Identifying $\mathbb{R}^4$ with $\mathbb{C} \times \mathbb{C}$, how would you write a_1 and a_2?

c) The maps $u \mapsto h(u,v) = (\cos u)a_1(v) + (\sin u)a_2(v)$ parameterize great circles with unit speed, and these circles meet the two axes at right angles. (What does it mean for two curves to meet at a right angle?) In this sense, h represents a helicoid in $\mathbb{S}^3$.

d) Try to identify the image surface for $a = 0$. What is the set where h_0 fails to be an immersion and what is its image?

e) Consider $a = 1$. What is the speed $\|\frac{\partial}{\partial v}h\|$? Find *periods* for h, that is, $(c,d) \neq (0,0)$ with minimal length such that $h(u+c, v+d) = h(u,v)$. Use a (non-rigorous) orientability argument to determine the topological type of the image surface.

Problem 10 – KLEIN BOTTLE:

We use the following which has not been defined formally in class: The Klein bottle is a non-orientable manifold, obtained by identifying opposite edges of a square: One pair of opposite edges in the same direction, the other in opposite directions.

a) Reason geometrically why the Klein bottle cannot be embedded into $\mathbb{R}^3$.
Hint: An embedding defines a continous normal.

b) Prove that the helicoid h_2 (or $h_{1/2}$) represents a Klein bottle immersed in $\mathbb{R}^4$. To do so, determine again minimal *periods* for h as in the previous problem.

c) Does h_2 represent an embedding of the Klein bottle into $\mathbb{S}^3$?

Problem 11 – CONTINUOUS IMAGE OF A SET OF MEASURE 0 WITH POSITIVE MEASURE:

Let $Q := [0,1] \times [0,1]$ be the square in the plane $\mathbb{R}^2$. A space-filling curve is a curve $c \colon [0,1] \to Q$ which is continuous and surjective. Use this example to construct a continous mapping $f \colon Q \to Q$ which maps a set of measure 0 to a set of positive measure.

Problem 12 – Matrices of fixed rank:

a) Check that the space of $m \times n$-matrices with rank 1 has the dimension stated in class. Is there any other rank, besides 0 and $\min(m,n)$, with an obvious dimension?

b) How many charts for M_r have we used in class to describe this submanifold of M?

c) Let $K \subset \mathbb{R}^n$ be compact, and $f\colon K \to \mathbb{R}^{n+k}$ with $k \geq 0$ be any differentiable map, i.e., f extends as a differentiable map to some open neighbourhood U of K. Find a matrix A, such that $x \mapsto Ax + f(x)$ has a Jacobian of rank n for all $x \in K$.

d) Try to characterize the boundary of M_r.

Problem 13 – Connectedness of the space of matrices with fixed rank:

a) Show that for $m = n$ the space of matrices with full rank has two components.

b) Prove that the space of 2×2 matrices of rank one is connected.

c) Prove that unless $r = m = n$ the space $\Phi^{-1}(0)$ is connected.

d) Prove that unless $r = m = n$ the space $\mathsf{M}_r(m \times n)$ is connected.

Problem 14 – Tangent space:

a) How did we define a tangent vector $v \in T_pM$ to a manifold M? What is the standard basis of T_pM with respect to a chart (x, U)?

b) Consider an implicitly defined submanifold $M = \varphi^{-1}(0)$, where φ has 0 as a regular value. How can you describe the tangent space?

c) If y is a chart which locally maps a submanifold $M \subset \mathbb{R}^{n+k}$ to a slice, i.e. $y(M \cap U) = y(U) \cap (\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}^k$, where $U \subset \mathbb{R}^{n+k}$, how would you describe the tangent space of M at $p \in M$?

Problem 15 – Grassmannians:

We consider

$$G(k,n) := \{k - \text{dimensional subvectorspaces } V \subset \mathbb{R}^n\}.$$

We want to prove that $G(k,n)$ is a manifold with a suitable differentiable structure.

a) Consider $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$. Prove that $U := \{V \in G(k,n) : V \cap (\{0\} \times \mathbb{R}^{n-k}) = \{0\}\}$ is a manifold by regarding U as the set of graphs $\Gamma(A)$ of linear mappings $A\colon \mathbb{R}^k \to \mathbb{R}^{n-k}$. What is the dimension? (Perhaps the same works implicitly.)

b) Find charts on sets similar to U that cover $G(k,n)$.

c) Show that the transition maps are differentiable (this is harder).

d) Find a bijection from $G(k,n)$ to $G(n-k,n)$. Is it a diffeomorphism?

Problem 16 – DEFINITIONS:

Recall the following definitions:
- tangent vector
- vector field
- differentiable map and differential
- immersion and embedding (is an injective immersion an embedding?)
- submanifold

Part 2. Vector fields, flows and the Frobenius theorem

8. Lecture, Thursday 3.12.09

5. VECTOR FIELDS

Vector fields are essential objects in order to study differential manifolds. In this section, we will see vector fields in three different roles: As geometric vector fields, as directional (or Lie) derivatives, and as the generator of a flow.

5.1. Geometric vector fields.

Definition. A *(differentiable) vector field* X on a manifold M is a mapping $p \mapsto X(p) \in T_pM$, such that $p \mapsto \big(p, X(p)\big)$ is a differentiable map from M to TM. We denote the vector space of all vector fields on M by $\mathcal{V}(M)$.

Let (x, U) be a chart. Then the vector field X has a *principal part* $\xi \colon U \to \mathbb{R}^n$ w.r.t. (x, U). If $e_i(p) = [x^{-1}\big(x(p) + tb_i\big)]$ is the standard basis w.r.t. (x, U) then ξ can be used to represent X as

$$X(p) = \sum_{i=1}^{n} \xi^i(p) e_i(p) \qquad \text{for all } p \in U,$$

see (7). We claim X is differentiable if and only if ξ is a differentiable function on U. Indeed, if y is the chart of TM associated to (x, U), as in (9), then by definition the map $p \mapsto y\big(p, X(p)\big) = \big(x(p), \xi(p)\big)$ is differentiable from U to $\mathbb{R}^{2n}$, and so in particular the second vector $\xi(p)$ is differentiable.

Examples. 1. On the torus T^2, there is a nice basis of non-vanishing vector fields. On the other hand, it is a theorem that neither $\mathbb{S}^2$ nor a surface of genus $g \geq 2$ (a surface with more than one hole), carries a vector field without a zero.

2. On $\mathbb{R}^n$ we always use the atlas $\{\mathrm{id}, \mathbb{R}^n\}$. The assignment of a curve to a principal part, $\xi(p) \mapsto X(p) = [p + t\xi(p)]$, then is a canonical isomorphism. We consider this map an identification of principal parts and equivalence classes of curves, and so we write

$$[p + t\xi(p)] = \xi(p), \qquad p \in \mathbb{R}^n, \tag{19}$$

as in (8).

5.2. **Lie derivative.** Let us go back to tangent vectors. In $\mathbb{R}^n$, a curve c through p with tangent vector $\xi := c'(0) \in \mathbb{R}^n$ induces a directional derivative *[Richtungsableitung]* $\partial_\xi f(p)$ for $f\colon \mathbb{R}^n \to \mathbb{R}$, namely

$$\frac{d(f\circ c)}{dt}(0) = \sum_{i=1}^n \frac{d}{dt}c^i(0)\frac{\partial f}{\partial x_i}\big(c(0)\big) = \sum_{i=1}^n \xi^i \frac{\partial f}{\partial x_i}(p) =: \partial_\xi f(p). \tag{20}$$

On a manifold M, let us denote the set of all (differentiable) functions with $\mathcal{D}(M) := C^\infty(M,\mathbb{R})$. If $c\colon I \to M$ is a curve, then for $f \in \mathcal{D}(M)$ we can certainly differentiate $f\circ c\colon I \to \mathbb{R}$ to a generalized directional derivative:

Definition. For each $v = [c] \in T_pM$ the *Lie derivative* of $f \in \mathcal{D}(M)$ is given by

$$\partial_v f := \frac{d(f\circ c)}{dt}(0).$$

Other common notation for the Lie derivative includes $v(f)$, $L_v f$,

To show the Lie derivative of c is well-defined and depends only on the class $[c]$ we calculate in coordinates:

$$\frac{d(f\circ c)}{dt}(0) = \frac{d}{dt}\big(f\circ x^{-1}\circ x\circ c\big)(0) = \underbrace{d(f\circ x^{-1})_{x(p)}}_{\text{independent of } c}\ \underbrace{\frac{d}{dt}\big(x\circ c\big)(0)}_{\text{depends only on } [c]}\ . \tag{21}$$

The standard calculus rules, applied to the function $f\circ c$, give that the Lie derivative ∂_v is
- $\mathbb{R}$-linear in $\mathcal{D}(M)$, $\partial_v(cf+g) = c\,\partial_v f + \partial_v g$ $\forall c\in\mathbb{R}$, $f,g \in \mathcal{D}(M)$, and
- satisfies the *product rule* $\partial_v(fg) = f\,\partial_v g + (\partial_v f)g$, $\forall f,g\in\mathcal{D}(M)$.

An operator with these properties is called a *derivation*. It is possible to introduce tangent vectors as derivations.

Let us give local representations of the Lie derivative. Denote the partial derivative w.r.t. the i-th coordinate in $\mathbb{R}^n$ by ∂_i. Then the right hand side of (21) reads:

$$\partial_v f = \sum_{i=1}^n \partial_i(f\circ x^{-1})\Big|_{(x\circ c)(0)} \frac{d}{dt}(x\circ c)^i(0).$$

In order to make the notation for manifolds appear as for the euclidean case, we set

$$\frac{\partial f}{\partial x_i}\Big|_p := \partial_i(f\circ x^{-1})\Big|_{x(p)} \quad\text{and}\quad \frac{\partial}{\partial x_i} := \partial_{e_i}. \tag{22}$$

This notation ignores the chart x and so lets the manifold formula become indistinguishable from the Euclidean formula (20):

$$\partial_v f(p) = \sum_{i=1}^n \xi^i\,\frac{\partial f}{\partial x_i}(p) \quad\text{or}\quad \partial_v = \sum_{i=1}^n \xi^i\,\frac{\partial}{\partial x_i} \qquad \text{where } \xi \text{ is the principal part of } v. \tag{23}$$

Example. The following calculation verifies that the Lie derivative agrees for different choices of charts:

$$\sum_{i=1}^{n} \xi^i \partial_i (f \circ x^{-1})\big|_{x(p)} = \sum_{i=1}^{n} \xi^i \partial_i (f \circ y^{-1} \circ y \circ x^{-1})\big|_{x(p)}$$
$$\overset{\text{chain rule}}{=} \sum_j \Big(\partial_j (f \circ y^{-1})\big|_{y(p)} \sum_i \xi^i \partial_i (y \circ x^{-1})^j\big|_{x(p)} \Big) = \sum_{j=1}^{n} \eta^j \, \partial_j (f \circ y^{-1})|_{y(p)}$$

Employing (22) the notation becomes

$$\sum_{i=1}^{n} \xi^i \frac{\partial f}{\partial x_i} \overset{\text{chain rule}}{=} \sum_{i,j} \xi^i \frac{\partial f}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_j \Big(\frac{\partial f}{\partial y_j} \sum_i \xi^i \frac{\partial y_j}{\partial x_i} \Big) = \sum_{j=1}^{n} \eta^j \frac{\partial f}{\partial y_j}. \tag{24}$$

We now consider the Lie derivative for vector fields. If $X \in \mathcal{V}(M)$ and $f \in \mathcal{D}(M)$ then $\partial_X f$ is defined at each point $p \in M$ and so $\partial_X f \in \mathcal{D}(M)$ again.

Locally, in terms of a chart x,

$$(\partial_X f)(p) = \sum_{i=1}^{n} \xi^i(p) \frac{\partial f}{\partial x_i}(p) \qquad \text{for } f \in \mathcal{D}(M).$$

∂_X is a differential operator of first order with the following properties:

1. ∂_X is an $\mathbb{R}$-linear map from $\mathcal{D}(M)$ to $\mathcal{D}(M)$, satisfying the product rule.
2. $(\partial_X f)(p)$ depends only on $X(p)$, and on f in a neighbourhood of p.
3. In $\mathbb{R}^n$ the directional derivative and differential are related by $df X = \partial_X f$. Similarly, we claim for the Lie derivative on manifolds:

$$df X = \partial_X f \qquad \text{for all } X \in \mathcal{V}(M),\ f \in \mathcal{D}(M). \tag{25}$$

To verify this, represent $X_p = [c]$ where $c\colon I \to M$, and observe that $f \circ c\colon I \to \mathbb{R}$ is a curve in $\mathbb{R}$. Given that the target of f is $\mathbb{R}$, we invoke the identification (19): The tangent vector $[f \circ c]$ is identified with its principal part which is $\frac{d}{dt}(f \circ c)(0)$. This gives, as desired:

$$df[c] \overset{\text{def. differential}}{=} [f \circ c] \overset{(19)}{=} \frac{d}{dt}(f \circ c)(0) \overset{\text{def. Lie deriv.}}{=} \partial_{[c]} f$$

4. Let $X, Y \in \mathcal{V}(M)$ and suppose $\partial_X f \equiv \partial_Y f$ holds for all $f \in \mathcal{D}(M)$. Then $X = Y$. To see this, pick a chart (x, U) and note that locally $\partial_X f = \sum \xi^i \frac{\partial f}{\partial x_i}$ and so $f := x^i \in \mathcal{D}(U)$ is a locally defined function with $\partial_X x^i = \xi^i$ for each i. Thus $\xi^i = \eta^i$ for each i and so $X = Y$.

5.3. **Flows of vector fields.** Each vector field on $\mathbb{R}^n$ defines a first order ordinary differential equation. We will see that the same problem is well-posed and solvable on manifolds.

Given $X \in \mathcal{V}(M)$ and $p \in M$, we want to determine a curve $c(t)\colon I \to M$, where $I \ni 0$ is an open (time) interval, such that c solves the initial value problem

$$c(0) = p, \quad \frac{d}{dt}c(t) = X\big(c(t)\big) \quad \text{for all } t \in I. \tag{26}$$

Such a c is called an *integral curve* of X through p. In the language of ODE's, our field X is not time-dependent, and so our ODE is autonomous.

Theorem 21. *For each $X \in \mathcal{V}(M)$ and all $p \in M$ there exists a solution to the initial value problem* (26). *It is unique in the sense that two such solutions agree where they are defined.*

Proof. Let $p \in M$ and consider a chart (x, U) at p with $x(U) =: \Omega \subset \mathbb{R}^n$. For the principal part $\xi(p)$ of $X(p)$ we have

$$\frac{d}{dt}(x \circ c)(t) = \xi\big(c(t)\big) = (\xi \circ x^{-1} \circ x \circ c)(t), \tag{27}$$

where c is to be determined. Writing $\tilde{\xi} := \xi \circ x^{-1}$ for a smooth vector field on Ω, and $\gamma := x \circ c$ for the chart representation of the desired curve in Ω we have

$$\frac{d}{dt}\gamma(t) = \tilde{\xi}\big(\gamma(t)\big), \qquad \gamma\colon I \to \Omega, \ \ \gamma(0) = x(p). \tag{28}$$

Local solutions γ of this ordinary differential equation on $U \subset \mathbb{R}^n$ are provided by the theorem of Picard-Lindelöf on some interval I containing 0, and so (26) can also be solved locally in time.

Any two solutions of (28) in Ω are unique, and so the solutions are independent of the chart chosen. Moreover, the uniqueness extends to the entire intersection of the time intervals of two solutions since we can appeal to the uniqueness of the intitial value problem also in any other chart. $\square$

`9. Lecture, Thursday 10.12.09`

Let us now change our point of view. Consider a moving fluid. The position of a particle p after time t defines a map $\varphi(t, p)$. There is a velocity field $X = \frac{d}{dt}\varphi(t, p)$, whose integral curves give the orbit (integral curve) of each particle $t \mapsto \varphi(t, p)$. That is, differentiating a flow gives a vector field, and integrating a vector field gives a flow. This concept works on manifolds:

Definition. A (local) *flow [Fluss]* on a manifold M is a differentiable mapping $\varphi\colon D \subset \mathbb{R} \times M \to M$, denoted $\varphi_t(p) = \varphi(t, p)$. Here $D \subset \mathbb{R} \times M$ is required to be open, containing $\{0\} \times M$, and for all $p \in M$ the set $D \cap (\mathbb{R} \times \{p\})$ must be an open interval. Moreover, φ must satisfy

$$\varphi_0 = \mathrm{id} \qquad \text{and} \qquad \varphi_{s+t} = \varphi_s \circ \varphi_t \quad \text{whenever defined.} \tag{29}$$

We need some more terminology:
• We call a flow $\varphi\colon D \to M$ *maximal* if φ does not admit an extension to any proper open superset $D' \supset D$.
• φ is *global* for X if φ is defined on all of $\mathbb{R} \times M$.
If M is compact then a maximal flow φ can be shown to be global ($\rightsquigarrow$ problems).

By setting $X(p) := \frac{d}{dt}\varphi_t(p)|_{t=0}$ we obtain a vector field on M. Conversely, given X a flow is obtained by the union of all integral curves of X:

Theorem 22. *Given a vector field X on a manifold M there is a unique maximal flow $\varphi\colon D \to M$, such that*

$$\varphi_0 = \mathrm{id} \qquad \textit{and} \qquad \frac{d}{dt}\varphi_t(p) = X\big(\varphi_t(p)\big) \qquad \textit{for all } p \in M. \tag{30}$$

Moreover, if φ is defined on all of $\mathbb{R} \times M$ each φ_t is a diffeomorphism onto its image.

If φ_t is defined on $(a, b) \times M$ it is also called a *local 1-parameter group of diffeomorphisms.*

Proof. We define $\varphi_t(p) := c(t)$ where c solves the initial value problem (26). By ODE theory we have the following properties:
• Since φ is maximal it is unique, by Thm. 21.
• Continuous dependence on initial conditions implies D is open.
• For X differentiable, the ODE solutions depend differentiably on initial conditions, meaning that φ depends differentiably on p

To verify $\varphi_{s+t} = \varphi_s \circ \varphi_t$ note that $t \mapsto \varphi_{s+t}(p)$ is the integral curve of X through $q := \varphi_s(p)$. Indeed $\varphi_{s+0}(p) = q$, and $\frac{d}{dt}\varphi_{s+t} = \frac{d}{d\tau}\varphi_\tau|_{\tau=s+t} = X(\varphi_{s+t})$ by the chain rule.

Let us prove φ_t is a diffeomorphism. We set $s = -t$ in $\varphi_{s+t} = \varphi_s \circ \varphi_t$ to see $\varphi_{-t} = (\varphi_t)^{-1}$ if defined. But $t \mapsto \varphi_{-t}$ is the flow of the vector field $-X$ as $\frac{d}{dt}\varphi_{-t} = -\frac{d}{d\tau}\varphi_\tau|_{\tau=-t} = -X(\varphi_{-t})$. The uniqueness theorem for φ_{-t} proves that φ_t is injective. Thus $\varphi_t(p) = q$ if and only if $\varphi_{-t}(q) = p$. Consequently, φ_{-t} is defined on $\varphi_t(M)$, and moreover φ_t and φ_{-t} are differentiable. $\square$

Examples. 1. The field $e_i \in \mathcal{V}(\mathbb{R}^n)$ defines the global flow $\varphi_t(p) = p + te_i$. Note that if we remove a point from $\mathbb{R}^n$, for instance the origin, the flow will no longer be global (determine D then!).
2. For $M = \mathbb{R}^2$ consider the vertical field $X(u, v) := (0, u)$. Then $\varphi_t(u, v) = (u, v + ut)$. Indeed,

$$\varphi_0(u, v) = (u, v) \quad \text{and} \quad \frac{d}{dt}\varphi_t(u, v) = \frac{d}{dt}(u, v + ut) = (0, u) = X\big(\varphi_t(u, v)\big).$$

2. The field $J(u,v) := (-v,u)$ on $\mathbb{R}^2$ has circles as integral curves, and $\varphi_t \in \mathsf{SO}(2)$ is a rotation by an angle t (verify!).
3. For $X \in \mathcal{V}(\mathbb{R}^n)$, we have the expansion $\varphi_t(p) = p + tX(p) + O(t^2)$ at $t = 0$ (problems).

6. Commuting flows and the Lie bracket

A single vector field has one-dimensional integral curves. In this section, we deal with a higher dimensional generalization: Suppose we have two or more vector fields:
• Can we integrate the fields to a surface or submanifold such that they become its coordinate vector fields (i.e., standard vector fields)?
• More generally, can we find a surface or integral submanifold which is the linear hull of the given fields?

We will answer both questions in terms of the Lie bracket.

6.1. **The Lie bracket of vector fields.** For $f \in \mathcal{D}(M)$ the Lie derivative $\partial_X f$ is again in $\mathcal{D}(M)$. Thus we can compose Lie derivatives: $\partial_X(\partial_Y f) \in \mathcal{D}(M)$. Locally, for $X = \sum_i \xi^i e_i$ and $Y = \sum_j \eta^j e_j$, this second Lie derivative is represented as

$$\partial_X(\partial_Y f) = \partial_X\Big(\sum_j \eta^j \frac{\partial f}{\partial x_j}\Big) = \sum_{i,j} \xi^i \frac{\partial \eta^j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum_{i,j} \xi^i \eta^j \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{31}$$

Consequently, $\partial_X\partial_Y$ involves second derivatives. On the other hand, $(\partial_Z f)(p)$ depends only on the first derivatives of f at p. Thus $\partial_X\partial_Y \neq \partial_Z$ for any $Z \in \mathcal{V}(M)$. However, the last assertion becomes true for a difference of second Lie derivatives:

Theorem 23. *Let X and Y be vector fields on a manifold M. Then there is a unique vector field $Z \in \mathcal{V}(M)$ such that $\partial_Z f := (\partial_X\partial_Y - \partial_Y\partial_X)f$ holds for all $f \in \mathcal{D}(M)$.*

Proof. Using (31) and the fact that second partials commute (Schwarz), we obtain with respect to a chart (x,U)

$$\partial_X\partial_Y f - \partial_Y\partial_X f = \sum_{i,k}\left(\xi^i \frac{\partial \eta^k}{\partial x_i} - \eta^i \frac{\partial \xi^k}{\partial x_i}\right)\frac{\partial f}{\partial x_k} \qquad \text{for all } p \in U \text{ and } f \in \mathcal{D}(M). \tag{32}$$

So on U our claim holds for

$$Z(p) := \sum_k \zeta^k(p) e_k(p) \quad \text{with principal part } \zeta^k := \sum_i \Big(\xi^i \frac{\partial \eta^k}{\partial x_i} - \eta^i \frac{\partial \xi^k}{\partial x_i}\Big).$$

But for each f, the iterated Lie derivative $(\partial_X\partial_Y - \partial_Y\partial_X)f$ is defined independently of the charts chosen, and so in fact $\partial_Z f$ is defined globally. Thus (32) does not depend on the chart (x,U), and Z is a (global) vector field on M. (You might as well convince yourself that the coefficients ζ^k transform with the Jacobian of the transition map – check this!). $\square$

We write $[X,Y]$ for the vector field Z and call it the *Lie bracket [Lie-Klammer]* or the *commutator [Kommutator]* of X and Y. Then

$$\partial_{[X,Y]} := \partial_X \partial_Y - \partial_Y \partial_X. \tag{33}$$

Examples. 1. Consider a chart (x,U). Then the standard basis e_i defines vector fields with constant principal parts $\xi^i = \delta^i_j$. Hence their commutator vanishes: $\big[e_i, e_j\big] = 0$.

2. For $M = \mathbb{R}^2$ we identify tangential vectors with principal parts. Let us consider the two fields

$$X(u,v) := (0,u) = \big(\xi_1(u,v), \xi_2(u,v)\big) \qquad Y(u,v) := (1,0) = \big(\eta_1(u,v), \eta_2(u,v)\big). \tag{34}$$

The only non-vanishing partial of the principal parts ξ_i, η_i is $\frac{\partial}{\partial u}\xi_2 = 1$. Hence

$$[X,Y] = (-\eta_1 \frac{\partial}{\partial u}\xi_2)e_2 = (-1 \cdot 1)e_2 = -(0,1).$$

Definition. A *Lie algebra* is an $\mathbb{R}$-vector space $\mathcal{A}$ with an $\mathbb{R}$-bilinear map $[\cdot,\cdot]\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ such that for all $X, Y, Z \in \mathcal{A}$ the following holds:
(i) Anticommutativity $[X,Y] = -[Y,X]$,
(ii) *Jacobi identity* $\big[[X,Y],Z\big] + \big[[Y,Z],X\big] + \big[[Z,X],Y\big] = 0.$

Examples. 1. $\mathbb{R}^3$ with the cross product,
2. $n \times n$ matrices with $[A,B] := AB - BA$, or Lie subalgebras (subvectorspaces closed under the bracket) like skew-Hermitian matrices,
3. (trivial): Any vector space with $[v,w] := 0$.
4. If $\mathcal{A}$ is one-dimensional then the Lie bracket must vanish.
5. A *Lie group* M is a manifold with a group structure. The tangent space T_eM to a Lie group at the identity $e \in M$ is endowed with the structure of a finite dimensional Lie algebra. For instance $\mathsf{SO}(n)$ is a Lie group and $T_E\mathsf{SO}(n) = \{\text{skew symmetric matrices}\}$ is a Lie algebra.
6. Any vector space $\mathcal{A}$ which is an associative algebra becomes a Lie algebra with $[a,b] := ab - ba$. Anticommutativity is obvious, the Jacobi identity is a short calculation.

`10. Lecture, Thursday 17.12.09` ______________________________

We can now state for the Lie bracket:

Theorem 24. *The vector fields $\mathcal{V}(M)$ with $[.,.]$ defined by (33) form a Lie algebra. Moreover, the Lie bracket satisfies*

$$[fX, gY] = fg[X,Y] + f(\partial_X g)Y - g(\partial_Y f)X \quad \textit{for all } f,g \in \mathcal{D}(M),\ X,Y \in \mathcal{V}(M). \tag{35}$$

We leave the proof of the formula for $[fX, gY]$ as an exercise.

6.2. **Commuting flows.** Suppose X, Y are two vector fields with flows φ_s, ψ_t, respectively. It is natural to ask whether $\varphi_s\psi_t(p) = \psi_t\varphi_s(p)$ holds, or equivalently $\mathrm{id} = \psi_{-t}\varphi_{-s}\psi_t\varphi_s$.

Examples. 1. Consider the two fields e_1, e_2 in $\mathbb{R}^2$ and the initial point 0. Let φ be the flow of e_1, and ψ be the flow of e_2. We follow the field e_1 for some time s, and the field e_2 for some time t, up to the point $\psi_t\varphi_s(0) = (s, t)$. On the other hand, we reach the same point by moving in opposite order, $\varphi_s\psi_t(0) = (s, t)$. We will relate this property to the fact $[e_1, e_2] = 0$.

2. For our earlier example from page 36 we have $\varphi_s(u, v) = (u, v + us)$ and $\psi_t(u, v) = (u + t, v)$. Hence

$$\psi_1\varphi_1(0, 0) = \psi_1(0, 0) = (1, 0) \qquad \neq \qquad \varphi_1\psi_1(0, 0) = \varphi_1(1, 0) = (1, 1).$$

We want to relate this fact to $[X, Y] \neq 0$.

We claim that flows commute if their generating vector fields have a vanishing Lie bracket.

Theorem 25. *Let $X, Y \in \mathcal{V}(M)$ be vector fields with flows φ, ψ, respectively. Then $\varphi_s\psi_t(p) = \psi_t\varphi_s(p)$ holds for all $p \in M$ and those s, t for which the equation is defined, if and only if $[X, Y] \equiv 0$.*

Thus the coordinate fields e_i, e_j on a manifold must have vanishing Lie bracket (the converse is true only locally). We need two lemmas for the proof.

Lemma 26. *Let X be a vector field on M and $p \in M$. If $X(p) \neq 0$, then there exists a chart (x, U) around p such that $X = e_1$ on U.*

Proof. Let (y, V) be a chart of M with $y(p) = 0$. Composing y with a rotation and dilation, we may assume the principal part of $X(p)$ points into the direction of the first basis vector of (y, V), meaning that $\xi(p) = b_1$. Let $\{b_1\}^\perp$ be the coordinate hyperplane. Then the restriction of y to the coordinate hyperplane $H = y^{-1}(\{b_1\}^\perp \cap y(V))$ remains a diffeomorphism.

We restrict the flow φ of X to a neighbourhood of $(0, p) \in \mathbb{R} \times H$, and write again φ for this map from time cross H to M. Then $\varphi_0 = \mathrm{id}|_H$ and so $d\varphi_{(0,p)}(0, v) = v$, meaning that tangent vectors v to H are preserved. On the other hand, $d\varphi_{(0,p)}(e_t, 0) = X_p \notin T_pH$ for e_t the unit vector in time. By the inverse mapping theorem, φ is a local diffeomorphism on some neighbourhood W of $(0, p) \in \mathbb{R} \times H$ to $U := \varphi(W) \subset M$.

Let us now define x in terms of φ and y: Since φ maps W to $U \subset \mathbb{R} \times H$ and y maps H to $\{b_1\}^\perp$ we set

$$x\colon U \to \mathbb{R} \times y(H) \subset \mathbb{R}^n, \qquad x := (\mathrm{id}_{\mathbb{R}}, y) \circ \varphi^{-1};$$

The inverse then is $x^{-1} = \big((\mathrm{id}_{\mathbb{R}}, y) \circ \varphi^{-1}\big)^{-1} = \varphi \circ (\mathrm{id}_{\mathbb{R}}, y^{-1})$.

In order to prove the claim we have to show that the tangents

$$[t \mapsto x^{-1}(u + tb_1)] = [t \mapsto \varphi(t + u_1, y^{-1}(u_2, \dots, u_n)] = [t \mapsto \varphi_t(\varphi_{u_1}(y^{-1}(u_2, \dots, u_n)))]$$

agree with X, which they do since φ is the flow of X. $\square$

Problem: Determine the map x for the rotation field $J(u, v) = (-v, u)$ on $\mathbb{R}^2 \setminus \{0\}$.

A vector field on $\mathbb{R}^n$ is a mapping $Y = (Y^1, \dots, Y^n)\colon \mathbb{R}^n \to \mathbb{R}^n$. The directional derivative of Y in direction of any other vector field X can be expressed in terms of the flow φ of X:

$$\partial_X Y(p) = \lim_{t\to 0} \frac{Y\big(\varphi_t(p)\big) - Y(p)}{t} = \frac{d}{dt}(Y \circ \varphi_t)\Big|_{t=0} = \sum \frac{d(Y \circ \varphi_t)^i}{dt}\Big|_{t=0} e_i\,, \tag{36}$$

using the fact $\varphi_t = \mathrm{id} + tX + O(t^2)$. On a manifold M, the same difference quotient is no longer meaningful: The vectors $Y\big(\varphi_t(p)\big) \in T_{\varphi_t(p)}M$ and $Y(p) \in T_pM$ are contained in different tangent spaces and so subtraction cannot be defined.

In order to use vector space operations on T_pM alone we need to move the first vector back to T_pM. Then we can assert:

Lemma 27. *If X, Y are vector fields on M, and φ is the flow of X, then*

$$[X, Y](p) = \lim_{t\to 0} \frac{d\varphi_{-t} Y(\varphi_t(p)) - Y(p)}{t} \qquad \text{for all } p \in M. \tag{37}$$

Problem: Confirm this formula for Example 2 on page 36.

Proof. Let us first consider the case $X(p) \neq 0$. According to Lemma 26 there is a chart (x, U) of M with $X = e_1$. With respect to this chart we have a constant local representation $X(q) = e_1(q)$ for $q \in U$, and so $\partial_{[X,Y]} = \partial_X \partial_Y$ holds in view of (32); here the right hand side is defined only in local coordinates.

The local representation of Y w.r.t. x is $Y(p) = \sum \eta^i(p) e_i(p)$. As long as defined, this gives $\varphi_t\big(x^{-1}(u)\big) = x^{-1}(u_1 + t, u_2, \dots, u_n)$, and thus $d\varphi_t(e_i(p)) = e_i(\varphi_t(p))$ for all t and $1 \le i \le n$. Consequently,

$$F\big(\varphi_t(p)\big) := d\varphi_{-t} Y(\varphi_t(p)) = \sum_i \eta^i\big(\varphi_t(p)\big) e_i(p).$$

The difference quotient in T_pM then verifies (37):

$$\begin{aligned}
\lim_{t\to 0} \frac{d\varphi_{-t} Y_{\varphi_t(p)} - Y_p}{t} &= \lim_{t\to 0} \frac{F\big(\varphi_t(p)\big) - F(p)}{t} \overset{(36)}{=} \sum_i \frac{d}{dt} \eta^i\big(\varphi_t(p)\big)\Big|_{t=0} e_i(p) \\
&= \sum_i \partial_X \eta^i(p) e_i(p) = \sum_i (\partial_X \eta^i(p) - \partial_Y \xi^i(p)) e_i(p) = [X, Y](p)
\end{aligned}$$

In case $X(p)$ vanishes identically on a neighbourhood of p, then $[X, Y](p) = 0$ on the one hand, and $\varphi_t = \mathrm{id}$ on the other hand, and so the right hand side of (37) vanishes. Finally, the case $X(p) = 0$, but $X(p_k) \neq 0$ for a sequence $p_k \to p$ results from the first case, by considering $k \to \infty$ and using continuity of our local representation of the difference quotient in p. $\square$

Proof of Thm. 25. "$\Longrightarrow$" Given the commuting property of the flows, the previous lemma saves us from the need to differentiate the flow equation twice. Indeed,

$$(38)\qquad \begin{aligned}(d\varphi_{-t})_p Y\big(\varphi_t(p)\big) &= d\varphi_{-t}\frac{d}{ds}\psi_s\big(\varphi_t(p)\big)\Big|_{s=0} \overset{\text{chain rule}}{=} \frac{d}{ds}\Big[\big(\varphi_{-t}\circ\psi_s\circ\varphi_t\big)(p)\Big]_{s=0}\\ &\overset{\text{assumption}}{=} \frac{d}{ds}\psi_s(p)\Big|_{s=0} = Y(p),\end{aligned}$$

and so $[X, Y] = 0$ by Lemma 27.

"$\Longleftarrow$" Here we must integrate our condition on the vector fields. Instead, we appeal to the uniqueness assertion of the Picard-Lindelöf theorem.

Let $Z(t) := d\varphi_{-t}Y(\varphi_t(p))$. We prove $\frac{d}{dt}Z(t) = 0$ for t small.

$$(39)\qquad \begin{aligned}\frac{d}{d\tau}Z(t+\tau)\Big|_{\tau=0} &= \frac{d}{d\tau}d\varphi_{-t-\tau}Y(\varphi_{t+\tau}(p))\Big|_{\tau=0} = \frac{d}{d\tau}d\varphi_{-t}d\varphi_{-\tau}Y\big(\varphi_{\tau+t}(p)\big)\Big|_{\tau=0}\\ &= d^2\varphi_{-t}\underbrace{\left(\frac{d}{d\tau}(d\varphi_{-\tau}\circ Y\circ\varphi_\tau)\Big|_{\tau=0}\right)}_{=0 \text{ by ass. \& Lemma 27}}\circ\varphi_t(p) = 0\end{aligned}$$

To see the last equality sign holds, note that the differential $d^2\varphi_{-t}$ of the linear map $d\varphi_{-t}$ is again a linear map, and so maps 0 to 0. Therefore, $Z(t)$ must be constant, which means $Z(0) = Y(p)$ equals $Z(t) = d\varphi_{-t}Y(\varphi_t(p))$.

It follows from (38) that for fixed t, the vector field $d\varphi_{-t}Y(\varphi_t(p))$ has the flow $s \mapsto \varphi_{-t}\circ\psi_s\circ\varphi_t$. Together with the last results this gives that Y has the flow ψ_s as well as the flow $s \mapsto \varphi_t\circ\psi_s\circ\varphi_{-t}$. But the local flow is unique, and so $\psi_s = \varphi_t\circ\psi_s\circ\varphi_{-t}$ which is the claim. $\square$

`11. Lecture, Thursday 14.1.10` ________________________________

6.3. **Frobenius theorem.** We now generalize integral curves to integral surfaces or manifolds:

Definition. (*i*) An n-dimensional *distribution* Δ on a manifold M^{n+k} is a mapping $p \mapsto \Delta(p) \subset T_pM$, where $\Delta(p)$ is an n-dimensional subspace. Here, the assignment must be smooth in the sense that each point p has a neighbourhood U and n vectorfields $X_1, \dots, X_n$ exist on U which span Δ at each point $p \in U$.
(*ii*) An n-dimensional submanifold $N \subset M$ is called an *integral manifold* of Δ if the inclusion map $i\colon N \to M$ satisfies $di_p(T_pN) = \Delta(p)$.

(iii) An n-dimensional distribution Δ is called *(locally) integrable*, if each $p \in M$ is contained in an integral submanifold of Δ.

Examples. 1. A nonvanishing vector field defines a one-dimensional distribution. The integral manifolds are integral curves.
2. On the torus $\mathbb{R}^2/\mathbb{Z}^2$, a constant vector field with irrational slope defines a one-dimensional distribution. It is locally integrable: The integral manifolds are lines of irrational slope. However, this works only locally, since globally an irrational line is not a submanifold. That is, there may not be a maximal integral submanifold.
3. Integral submanifolds need not exist, not even locally. A simple example is a 2-plane distribution in $\mathbb{R}^3$, spanned by $X(p) := e_1$ and $Y(p) := e_2 + p^1 e_3$.

We will relate integrability to the following. We call a distribution Δ *involutive* if for $X, Y \in \mathcal{V}(M)$ such that $X(p), Y(p) \in \Delta(p)$ for all $p \in M$ also $[X,Y](p) \in \Delta(p)$.

Example. For $X(p) := e_1$ and $Y(p) := e_2 + p^1 e_3$ we have $[X,Y] = e_3$, and so the distribution $\Delta(p) = \operatorname{span}\{X(p), Y(p)\}$ is not involutive.

It is enough to check involutiveness on a basis:

Lemma 28. *Suppose each $p \in M^{n+k}$ has a neighbourhood U such that $X_1, \dots, X_n \in \mathcal{V}(U)$ span Δ and $[X_i, X_j](p) \in \Delta(p)$ for all $1 \le i, j \le n$ then Δ is involutive.*

Proof. This is a linear algebra fact: If $X = \sum \xi^i X_i$ and $Y = \sum \eta^j X_j$ then indeed

$$[X,Y] = \sum_{i,j} [\xi^i X_i, \eta^j X_j] \overset{(35)}{=} \sum_{i,j} \Big(\xi^i \eta^j [X_i, X_j] + \xi^i \partial_{X_i} \eta^j X_j - \eta^j \partial_{X_j} \xi^i X_i \Big) \in \Delta \qquad \square$$

We need some preparatory notions and lemmas. To calculate Lie brackets, it is useful to know how the Lie bracket transforms under a differentiable map $\varphi \colon M \to \tilde{M}$. We call $X \in \mathcal{V}(M)$ and $\tilde{X} \in \mathcal{V}(\tilde{M})$ *φ-related [φ-verwandt]*, if

$$d\varphi(X) \equiv \tilde{X} \circ \varphi.$$

Note that the integral curves of φ-related vector fields are related as images under φ.

Suppose a curve $c(t)$ represents X at p, that is, $X(p) = [c]$. Then Lie derivatives are easily related, namely for all $f \in \mathcal{D}(\tilde{M})$

$$\begin{aligned}(\partial_{\tilde{X}} f)(\varphi(p)) &= \partial_{\tilde{X}(\varphi(p))} f = \partial_{d\varphi[c]} f \overset{\text{def. differential}}{=} \partial_{[\varphi \circ c]} f \\ &\overset{\text{def. Lie der.}}{=} \frac{d}{dt}(f \circ \varphi \circ c)(0) \overset{\text{def. Lie der.}}{=} \partial_{[c]}(f \circ \varphi) = \partial_X (f \circ \varphi)(p).\end{aligned}$$

and so

$$(\partial_{\tilde{X}} f) \circ \varphi = \partial_X (f \circ \varphi). \tag{40}$$

The following result is no surprise in view of the fact that the Lie bracket measures the extend to which the flows of two vector fields commute:

Lemma 29. *Suppose $X, Y \in \mathcal{V}(M)$ are φ-related to $\tilde{X}, \tilde{Y} \in \mathcal{V}(\tilde{M})$. Then $[X,Y]$ is φ-related to $[\tilde{X}, \tilde{Y}]$, that is, $[\tilde{X}, \tilde{Y}](\varphi(p)) = d\varphi_p[X,Y] =: \widetilde{[X,Y]}(p)$*

Proof. For any $f \in \mathcal{D}(\tilde{M})$ we have:

$$\begin{aligned}(\partial_{[\tilde{X},\tilde{Y}]} f) \circ \varphi &= \big(\partial_{\tilde{X}}(\partial_{\tilde{Y}} f)\big) \circ \varphi - \big(\partial_{\tilde{Y}}(\partial_{\tilde{X}} f)\big) \circ \varphi \\ &\overset{(40)}{=} \partial_X\big((\partial_{\tilde{Y}} f) \circ \varphi\big) - \partial_Y\big((\partial_{\tilde{X}} f) \circ \varphi\big) \overset{(40)}{=} \partial_X\big(\partial_Y(f \circ \varphi)\big) - \partial_Y\big(\partial_X(f \circ \varphi)\big) \\ &= \partial_{[X,Y]}(f \circ \varphi)\end{aligned}$$

Consequently

$$df[\tilde{X}, \tilde{Y}] \circ \varphi = df(d\varphi[X,Y])$$

which means, as desired, $[\tilde{X}, \tilde{Y}] \circ \varphi = d\varphi[X,Y]$. □

We generalize Lemma 26 now to several vector fields.

Proposition 30. *Let $X_1, \ldots, X_n$ be linearly independent vector fields on a $n+k$-dimensional manifold M^{n+k}, defined in a neighbourhood of a point p. Suppose that on this neighbourhood, $[X_i, X_j] \equiv 0$ for $1 \le i, j \le n$. Then there is a coordinate system (x, U) around p with standard basis e_j, such that $X_i = e_i$ on U for $i = 1, \ldots, n$.*

Proof. Let us first assume $M = \mathbb{R}^{n+k}$, and $p = 0$, as well as $X_j(0) = b_j$ for $j = 1, \ldots, n$. Suppose φ^j is the flow generated by X_j, and define

$$\chi(u^1, \ldots, u^{n+k}) := \varphi^1_{u^1}\Big(\varphi^2_{u^2}\Big(\cdots\Big(\varphi^n_{u^n}\big(0, \ldots, 0, u^{n+1}, \ldots, u^{n+k}\big)\Big)\cdots\Big)\Big),$$

where u are the coordinates or $\mathbb{R}^{n+k}$. Then $d\chi_0(b_i) = \mathrm{id}$ since

$$d\chi_0(b_i) = \begin{cases} X_i(0) = b_i & i = 1, \ldots, k, \\ b_i & i = n+1, \ldots, n+k. \end{cases}$$

Hence $x := \chi^{-1}$ is a chart in some neighbourhood of $p = 0$.

We have $X_1 = e_1$ since the curves $u + tb_1$ have tangent vector X_1. We now use our hypothesis on the Lie bracket to prove the same for the indices from 2 to n. By Thm. 25 the hypothesis allows us to write

$$\chi(u^1, \ldots, u^{n+k}) := \varphi^j_{u^j}\Big(\varphi^1_{u^1}\Big(\cdots\Big(\varphi^n_{u^n}\big(0, \ldots, 0, u^{n+1}, \ldots, u^{n+k}\big)\Big)\cdots\Big)\Big),$$

and so as before we have $X_j = e_j$ for $j = 1, \ldots, n$ as well. □

Theorem 31. *An involutive n-dimensional distribution Δ on a manifold M^{n+k} is integrable. More precisely, for each $p \in M^{n+k}$ there is a chart (x, U) such that for each $a \in \mathbb{R}^k$ the set $N(p, a) := \{q \in U : (x^{n+1}(q), \dots x^{n+k}(q)) = a\}$ is an integral submanifold, and $p \in N(p, 0)$.*

Proof. Let us first prove the theorem for $M = \mathbb{R}^{n+k}$. By a motion, we can assume $p = 0$ and $\Delta(0) = \mathbb{R}^n \times \{0\}$. Denote with $\pi = d\pi \colon \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection onto the first n components. Since $d\pi_0$ restricted to $\Delta(0)$ is an isomorphism to $\mathbb{R}^n$, by continuity there is a neighbourhood U of 0, such that the restriction $d\pi \colon \Delta(q) \to \mathbb{R}^n$ is bijective for all $q \in U$. Hence the preimage of the standard basis defines vector fields $X_1, \dots, X_n \in \Delta(q)$ with $d\pi(X_i) = e_i$ for $i = 1, \dots, n$. That is, the vector fields e_i and X_i are π-related. By Lemma 29,

$$d\pi([X_i, X_j](q)) = [e_i, e_j](\pi(q)) = 0.$$

Since $d\pi$ is an isomorphism on $\Delta(q) = \operatorname{span}\{X_1(q), \dots, X_n(q)\}$ this implies $[X_i, X_j] = 0$ for all $i, j \le n$.

Hence we can apply Prop. 30 to obtain a coordinate system (y, U) such that the X_i become the standard basis. Then for each $a \in \mathbb{R}^k$, the sets $\{q \in U : y^{n+1} = a^1, \dots, y^{n+k} = a^k\}$ are integral manifolds.

To obtain the result for a manifold, consider an arbitrary chart x and compose it with the above chart y. $\square$

6.4. **Problems.**

Problem 17 – VECTOR FIELDS AND DIVISION ALGEBRAS:

Assume that on some $\mathbb{R}^n$ there is the structure of a *division algebra*, that is, a bilinear map $\beta \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, written as $(x, y) \mapsto xy$, such that all maps

$$\lambda_x \colon \mathbb{R}^n \to \mathbb{R}^n, \quad y \mapsto xy \qquad \text{and} \qquad \rho_y \colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto xy$$

are bijective. We do not assume that the multiplication β is associative, but we assume there is a unit element $e \in \mathbb{R}^n$ with $ex = xe = x$ for all $x \in \mathbb{R}^n$. Prove the following:

a) If $n > 1$ and $x \notin \mathbb{R}e$ then λ_x has no real eigenvalues.
   Hint: If $xy = \mu y$ then $(x - \mu e)y = 0$.
b) n is even. *Hint:* Recall a linear algebra result on eigenvalues.
c) We extend $b_n = e$ to a basis $(b_1, \dots, b_n)$ of $\mathbb{R}^n$ and consider the corresponding vector fields $X_j := X_{\lambda_{b_j}}$ for $j = 1, \dots, n$ on $\mathbb{S}^{n-1}$. Show that for each $x \in \mathbb{S}^{n-1}$, the vectors $X_1(x), \dots, X_{n-1}(x)$ are linearly independent.
   Hint: $\operatorname{span}\{x, b_1 x, \dots, b_{n-1} x\} = \rho_x(\mathbb{R}^n) = \mathbb{R}^n$.

d) An n-manifold is *parallelizable* if there are n vector fields which give a basis of each tangent space. Show that $\mathbb{S}^{n-1}$ is parallelizable if $\mathbb{R}^n$ carries the structure of a division algebra.

e) Show that the matrix group

$$\mathbb{H} := \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$$

gives $\mathbb{R}^4 = \mathbb{C}^2$ the structure of a four-dimensional associative division algebra, called quaternions.

Problem 18 – PREPARATION FOR LIE DERIVATIVES:

a) Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$. Define the directional derivative of f at $p \in \mathbb{R}^m$ with respect to a direction $\xi \in \mathbb{R}^m$.

b) Relate the directional derivative to the differential; state the result also with sums and indices, avoiding matrix notation.

Problem 19 – LIE BRACKET OF VECTOR FIELDS:

Prove $[fX, gY] = fg[X,Y] + f(\partial_X g)Y - g(\partial_Y f)X$ for all $f, g \in \mathcal{D}(M)$, $X, Y \in \mathcal{V}(M)$.
Hint: Calculate $\partial_{[fX,gY]}h$ for all $h \in \mathcal{D}(M)$.

Problem 20 – LIE SUBALGEBRAS:

a) An $n \times n$ matrix is skew-Hermitian if ${}^t\overline{A} = -A$. Prove that the set of skew-Hermitian matrices is closed under $[A, B] = AB - BA$.

b) Find another such matrix algebra. *Hint:* trace

Problem 21 – DETERMINATION OF FLOWS:

Determine the flow of the following vector fields on $\mathbb{R}^2$:

$$\text{a) } X\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2y \end{pmatrix}, \qquad \text{b) } Y\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}, \qquad \text{c) } Z\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Problem 22 – EXPANSION OF A FLOW:

For $X \in \mathcal{V}(\mathbb{R}^n)$ verify the expansion $\varphi_t(p) = p + tX(p) + O(t^2)$ at $t = 0$.

Problem 23 – FLOWS ON COMPACT MANIFOLDS:

Recall that a flow φ of a vector field $X \in \mathcal{V}(M)$ is global if $\varphi(t, p)$ exists for all $t \in \mathbb{R}$ and $p \in M$. Prove that the maximal flow φ is global if M is compact and $X \in \mathcal{V}(M)$.
Hint: On a compact manifold, each sequence has a convergent subsequence.

Problem 24 – INDEX OF A VECTOR FIELD ON A SURFACE:

Suppose the vector field $X \in \mathcal{V}(\mathbb{R}^2)$ has only a discrete set of zeros Z. For any differentiable loop (closed curve) $c(t)$ in $\mathbb{R}^2 \setminus Z$, define the number

$$i(X,c) := \frac{1}{2\pi} \int \varphi'(t)\, dt, \qquad \text{where } \varphi(t) := \angle(X(c(t)) \text{ is continuous,}$$

as the total change of angle along c which X makes against a constant vector field $E \neq 0$.

a) Prove that $i(X,c)$ does not depend on E.
b) Prove that loops c_1, c_2 which are (differentiably) homotopic in $\mathbb{R}^2 \setminus Z$ have the same index, $i(X,c_1) = i(X,c_2)$.
c) Let $p \in Z$ and c be a loop in $\mathbb{R}^2 \setminus Z$ which is null homotopic in $\{p\} \cup (\mathbb{R}^2 \setminus Z)$ and has winding number $+1$ about p. Then the *index* $j(X,p)$ of X at p is defined by $j(X,p) := i(X,c)$. (Compare with the beautiful pictures on p. 109 of Hopf's book: Differential Geometry in the Large)
d) If you attend Riemannian geometry: Note we defined the angle with respect to the standard Riemannian metric $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$. Prove that if g is any other Riemannian metric on $\mathbb{R}^2$, the similarly defined number $i(X,c) := i(g,X,c)$ agrees.
e) Extensions: Reason that $i(X,c)$ is defined on differentiable manifolds M. Do you have any idea for a similar number in higher dimensions?

Problem 25 – FLOWS AND LIE BRACKETS:

Consider $X(u,v) := (0,u)$ on $\mathbb{R}^2$.

a) Plot $X(u,v)$.
b) Find a chart $(x,U)\colon U \to \mathbb{R}^2$ around the point $(1,0)$ such that $X = e_1$, as in Lemma 26. Formulate this first as a condition on the differential $dx\colon \mathbb{R}^2 \to \mathbb{R}^2$. What is the maximal choice of U?
c) If you like: Discuss all choices for x. Remember to verify that x is a diffeomorphism.
d) Moreover, let $Y(u,v) := (1,0)$, see the example in class. Verify Lemma 27 at the point $(1,0)$.

Problem 26 – CYLINDRICAL COORDINATES:

On $\Omega := \mathbb{R}^3 \setminus \{(0,0,w) : w \in \mathbb{R}\}$ consider the vector fields $X(u,v,w) := \frac{1}{\sqrt{u^2+v^2}}(u,v,0)$ and $Y(u,v,w) := (J(u,v),0) = (-v,u,0)$.

a) Plot X and Y (the first two components!). Can you see what $[X,Y]$ is?
b) Verify they span an involutive distribution Δ.
c) Pick a point in Ω and determine a chart (x,U) as in Prop. 30.

Problem 27 – NON-INTEGRABLE DISTRIBUTION:

Check explicitely that $X(p) = e_1$ and $Y(p) = e_2 + p^1 e_3$ is non-integrable.

Part 3. Differential forms and Stokes' theorem

12. Lecture, Thursday 21.1.10

Stokes' theorem generalizes the fundamental theorem of calculus to several dimensions. It includes the classical integral theorems like the divergence theorem or Green's theorem. I prepared this part from Spivak's book [Spi]. It is worth to compare with a source which presents the theorem for submanifolds of $\mathbb{R}^n$ such as Forster's Analysis 3: The amount of technical work is no less in Euclidean space, and at some places the explicit calculations make the theory less transparent.

7. Differential forms

Let us motivate our approach with two examples.

1. Path integrals: Along a curve $c\colon I \to \mathbb{R}^n$ they may be introduced as $\int_I \langle X(c(t)), c'(t)\rangle\, dt$ for a vector field X. This integral is parameterization invariant by the chain rule:

$$\int_J \big\langle X(c(\varphi(s))), (c\circ\varphi)'(s)\big\rangle\, ds \;\; = \int_J \big\langle X(c(\varphi(s))), c'(\varphi(s))\big\rangle \varphi'(s)\, ds = \int_I \big\langle X(c(t)), c'(t)\big\rangle\, dt.$$

We know or will see that any 1-form ω defines such an integral more generally, namely $\int_c \omega := \int_I \omega(c'(t))\, dt$, which is parameterization invariant by the same calculation. Clearly, in this case, setting $X^i(p) := \omega_p(e_i)$ we can rewrite $\int_c \omega := \int_I \langle X, c'\rangle\, dt$.

2. Surface area in $\mathbb{R}^3$. The area of a rectangle spanned by v, w in $\mathbb{R}^3$ only depends on $v\times w = -w\times v$, that is, on an alternating 2-form of v, w. Similarly, the Gram determinant is alternating, for instance for a surface,

$$A(f) = \int \sqrt{\det df\, df^t} = \int \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2} = \int \sqrt{f_x \times f_y}.$$

Thus our first goal will be to introduce parameter independent alternating forms.

7.1. Multilinear algebra.

Let V be a real vector space with dual space V^*.

Definition. (i) A function $T\colon V^k \to \mathbb{R}$ is *k-multilinear* or a *k-tensor* if

$$v_i \mapsto T(v_1, \dots, v_i, \dots, v_k)$$

is linear for each i and all $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k \in V$.
(ii) T is *alternating* or *skew-symmetric* if for all $v_1, \dots, v_k \in V$

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \quad \forall\, 1 \le i \ne j \le k.$$

(iii) We denote the vector space of multilinear mappings by $\otimes^k V$, the subspace of alternating maps by $\Lambda^k V$.

We also define 0-tensors to be a real numbers, that is, $\otimes^0 V = \Lambda^0 V := \mathbb{R}$.

Examples. 1. $\otimes^1 V = V^*$,
2. $\otimes^2 V = \{\text{bilinear forms on } V\}$.
3. For $\dim V = n$ we have $\det \in \Lambda^n V$.

The following properties are equivalent to that T alternates:
• T vanishes if any pair of vectors coincides, $v_i = v_j$ for $i \neq j$ (by polarisation).
• $T\big(\sigma(v_1), \dots, \sigma(v_k)\big) = (\operatorname{sgn}\sigma) T(v_1, \dots, v_k)$ for all permutations $\sigma \in S^k$ and all $v_i \in V$ (why?).

To discuss basis representations, let us assume from now on that V has finite dimension. As we know, a bilinear form $b \in \otimes^2 V$ on a vector space V with basis (e_i) can be represented

$$b(v, w) = b\Big(\sum_i v^i e_i, \sum_j w^j e_j\Big) = \sum_{ij} b(e_i, e_j) v^i w^j = \sum_{ij} b_{ij} v^i w^j,$$

using multilinearity. In case b is alternating then b_{ij} is an antisymmetric matrix. Likewise, in general

$$T(v_1, \dots, v_k) = \sum_{i_1, \dots i_k} T_{i_1, \dots i_k} v^{i_1} \cdots v^{i_k}.$$

Clearly, if V has dimension n then the n^k coefficients $T_{\dots} \in \mathbb{R}$ show that $\dim(\otimes V) = n^k$.

If $S \in \otimes^k V$ and $T \in \otimes^l V$ then there is a *tensor product* $\otimes\colon \otimes^k V \times \otimes^l V \to \otimes^{k+l} V$,

$$\big(S \otimes T\big)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+l}) := S(v_1, \dots, v_k)\, T(v_{k+1}, \dots, v_{k+l}).$$

Suppose $e^1, \dots, e^n$ is a basis for V^*. Then the n^k tensors

$$\{e^{i_1} \otimes \cdots \otimes e^{i_k} : 1 \leq i_1, \dots, i_k \leq n\}$$

form a basis for $\otimes^k V$.

In case S and T are alternating, the tensor product $S \otimes T$ will in general not be alternating. For example, consider $S, T \in V^*$. Then the bilinear form $S \otimes T$ is not alternating, unless $v \mapsto S(v)T(v)$ vanishes identically. However,

$$S \wedge T := S \otimes T - T \otimes S$$

is alternating as $S(v)T(v) - T(v)S(v) \equiv 0$. Note that $S \wedge T$ vanishes if and only if S and T are linearly dependent.

The same construction works in general. In a first step, we define

$$\operatorname{Alt}\colon \otimes^k V \to \Lambda^k V, \qquad \operatorname{Alt} T(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\, T\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big).$$

Indeed, $\operatorname{Alt}(T)$ is alternating, and for T alternating $\operatorname{Alt}(T) = T$ (for this to hold we need $1/k!$). Hence we can consider Alt a projection map.

In a second step, we use Alt to define the *wedge product*

$$\wedge\colon \Lambda^k V \times \Lambda^l V \to \Lambda^{k+l} V, \qquad S \wedge T := \frac{(k+l)!}{k!\,l!}\operatorname{Alt}(S \otimes T)$$

Example. For $S, T \in \Lambda^1 V = V^*$ this gives, as before,

$$S \wedge T(v,w) = S \otimes T(v,w) - S \otimes T(w,v) = \big(S \otimes T - T \otimes S\big)(v,w).$$

In particular, $S \wedge T = -T \wedge S$, and $T \wedge T = 0$ holds for one-forms.

The wedge product has the following properties (problems?):
• Bilinearity: $(\omega, \eta) \mapsto \omega \wedge \eta$ is linear in each argument.
• Anticommutativity: If $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$ then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$. In particular, $\omega \wedge \omega = 0$ for k odd. (Exhibit an example $\omega \in \Lambda^2 V$ such that $\omega \wedge \omega \neq 0$.)
• Associativity: $(\omega \wedge \eta) \wedge \vartheta = \omega \wedge (\eta \wedge \vartheta)$. See [Sp] Thm. 2 of Ch. 7.
• Normalization: For the standard dual basis e^i of V we have $(e^1 \wedge \ldots \wedge e^n)(v_1, \ldots, v_n) = \det(v_1, \ldots, v_n)$, explaining the factors of the wedge product (see also [Sp], p. 279).

Most of these properties become much more obvious once we have exhibited a basis for $\Lambda^k V$:

Lemma 32. *If V has the basis $e^1, \ldots, e^n$ then*

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k} : 1 \le i_1 < \cdots < i_k \le n\} \tag{41}$$

is a basis for $\Lambda^k V$. In particular, $\dim \Lambda^k V = \binom{n}{k}$.

Permuted basis elements differ by at most a sign: $e_2 \wedge e_1 = -e_1 \wedge e_2$ or $e_3 \wedge e_1 \wedge e_2 = -e_1 \wedge e_3 \wedge e_2 = e_1 \wedge e_2 \wedge e_3$.

Proof. Linear independence: We consider a linear combination of elements of (41), evaluate it on a particular multivector, and claim

$$\sum_{j_1 < \ldots < j_k} a_{j_1,\ldots,j_k} e^{j_1} \wedge \cdots \wedge e^{j_k}\big(e_{i_1}, \ldots, e_{i_k}\big) = a_{i_1,\ldots,i_k}.$$

Here, (e_i) is a basis for V, dual to (e^j), and the basis elements of the multivector have increasing indices $1 \le i_1 < \ldots < i_k \le n$. Assuming our claim, a linear combination which vanishes can only have zero coefficients.

To prove the claim, we must evaluate the linear combination on our particular multivector. On the one hand, if an element of the linear combination has precisely the same index set, then

$$
\begin{aligned}
e^{i_1}\wedge\cdots\wedge e^{i_k}\big(e_{i_1},\dots,e_{i_k}\big) &= n!\,\mathrm{Alt}\,\big(e^{i_1}\otimes\cdots\otimes e^{i_k}\big)\big(e_{i_1},\dots,e_{i_k}\big)\\
&= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\big(e^{i_1}\otimes\cdots\otimes e^{i_k}\big)\big(e_{\sigma(i_1)},\dots,e_{\sigma(i_k)}\big)\\
&= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)e^{i_1}(e_{\sigma(i_1)})\cdots e^{i_k}(e_{\sigma(i_k)}) = (\operatorname{sgn}\mathrm{id})\,e^{i_1}(e_{\mathrm{id}(i_1)})\cdots e^{i_k}(e_{\mathrm{id}(i_k)}) = 1.
\end{aligned}
$$

On the other hand, any other basis element $e^{j_1}\wedge\cdots\wedge e^{j_k}$ in (41) has a different index set $\{j_1,\dots,j_k\}\neq\{i_1,\dots,i_k\}$, i.e., one index set is not a permutation of the other. Hence if we apply this other basis element to our given multivector $\big(e_{i_1},\dots,e_{i_k}\big)$ we obtain zero, as claimed.

Spanning property: We use the property that for any $\tau\in S_k$ we have $S_k=\{\tau\circ\sigma:\sigma\in S_k\}$. Hence by relabelling permutations we can assert:

$$
\begin{aligned}
\big(e^{\tau(i_1)}\wedge\cdots\wedge e^{\tau(i_k)}\big)\big(v_1,\dots,v_k\big) &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)e^{\tau(i_1)}(v_{\sigma(1)})\cdots e^{\tau(i_k)}(v_{\sigma(k)})\\
&= \sum_{\sigma\in S_k}\operatorname{sgn}(\tau^{-1}\circ\sigma)(\operatorname{sgn}\tau)e^{i_1}(v_{\tau^{-1}\circ\sigma(1)})\cdots e^{i_k}(v_{\tau^{-1}\circ\sigma(k)})\\
&= \operatorname{sgn}(\tau)\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)e^{i_1}(v_{\sigma(1)})\cdots e^{i_k}(v_{\sigma(k)}) = \operatorname{sgn}(\tau)\big(e^{i_1}\wedge\cdots\wedge e^{i_k}\big)\big(v_1,\dots,v_k\big).
\end{aligned}
$$

□

Example. 1. For $k=n$, we have $\dim\Lambda^nV=1$. Since $\det\in\Lambda^nV$, any n-form must be a scalar multiple of the determinant. For $k>n$, we have $\Lambda^kV=\emptyset$.

2. In $\mathbb{R}^3$ the form $e^1\wedge e^2$ is 1 on (e_1,e_2), but 0 on (e_2,e_3) and (e_1,e_3). By linearity, $e^1\wedge e^2(v_1,v_2)$ measures the projection of the parallelogram (v_1,v_2) to the e_1,e_2-plane (Proof: problems).

3. Let $a^1,a^2,b^1,b^2\in\mathbb{R}^{3*}$. Then there exist $c^1,c^2\in\mathbb{R}^{3*}$ such that $a^1\wedge a^2+b^1\wedge b^2=c^1\wedge c^2$. So the sum has a geometric interpretation as in the previous example.
indeed, any set of two non-equal planes in $\mathbb{R}^3$ shares a common vector. (Problem: Check this first for the special case $e_1\wedge e_2+e_2\wedge e_3$.Then prove the general statement.)

4. However, the 2-form $\omega:=e^1\wedge e^2+e^3\wedge e^4$ in $\mathbb{R}^4$, satisfies $\omega(e_1,e_2)=1$ and $\omega(e_3,e_4)=1$. It is beyond a classical area element in the sense that it cannot be expressed as $v\wedge w$; we say it is not *decomposable* ($\rightsquigarrow$ problems).

Remark. The previous example indicates that two-forms are linear combinations of "area elements". Similarly we can consider k-forms as the closure of k-dimensional area elements under vector space operations. It is a common idea in algebraic topology to take geometric objects and consider the algebraic closure of them.

`13. Lecture, Thursday 28.1.10` ______________________________

Problem. If $\omega_1, \ldots, \omega_k \in \Lambda^1 V$ then $\omega_1 \wedge \ldots \wedge \omega_k \equiv 0$ if and only the (ω_i) are linearly dependent.

Given a linear map $A\colon V \to W$, to a k-form ω on W we can associate the *pull-back of a k-form* which is the k-form $A^*\omega$ on V

$$A^* : \Lambda^k W \to \Lambda^k V\colon \qquad \omega \mapsto (A^*\omega)(v_1, \ldots, v_k) := \omega(Av_1, \ldots, Av_k)$$

The most important case are endomorphisms, $A\colon V \to V$, in particular coordinate transformations. For the determinant $\det(v_1, \ldots, v_n)$ which measures the oriented volume of the parallelepiped spanned by n vectors v_i, the change in volume amounts to $\det(Av_1, \ldots, Av_n) = \det A \det(v_1, \ldots, v_n)$. Since n-forms are constant multiples of the determinant, they must transform the same way:

Proposition 33. *Let $\omega \in \Lambda^n V$ and $A = (a_{ij})$ be an $n \times n$-matrix. Then*

$$\omega(Av_1, \ldots, Av_n) = \det A\, \omega(v_1, \ldots, v_n) \qquad \text{for all } v_1, \ldots, v_n \in V.$$

Proof. For fixed v_i's, the form

$$\text{columns of } A \mapsto \omega(Av_1, \ldots, Av_n)$$

is alternating and linear. Since $\dim \Lambda^n V = 1$ this must equal $c \det(\text{columns of } A)$ for all A. But setting A to the identity matrix, $A := E_n$, we obtain $c = c \det E_n = \omega(v_1, \ldots, v_n)$, which gives the desired formula. □

7.2. **Alternating forms on manifolds.**

Definition. Let M be a manifold. A map $p \mapsto \omega(p) \in \Lambda^k T_p M$ is a called a *k-form on M* if it is differentiable in the following sense: $(X_1, \ldots, X_k) \mapsto \omega(X_1, \ldots, X_k) \in \mathcal{D}(M)$ is differentiable whenever $X_1, \ldots, X_k \in \mathcal{V}(M)$. The set of all k-forms on M is denoted by $\Lambda^k M$; we set $\Lambda^0 M := \mathcal{D}(M)$; the set of all forms is denoted by $\Lambda M := \bigoplus_{k=0}^n \Lambda^k M$.

We have the following properties:
- $\Lambda^1 M = (\mathcal{V}(M))^*$, which is usually denoted as $\mathcal{V}^* M$.
- By their pointwise definition, k-forms are $\mathcal{D}(M)$-linear. That is, they are *tensor fields.*
- k-forms can be pulled back along $f\colon M \to N$ pointwise ($f^*\omega :=$ “$\omega \circ df$”).
- There is again a wedge product

$$\wedge\colon \Lambda^k M \times \Lambda^l M \to \Lambda^{k+l} M,$$

defined pointwise. It is bilinear, anticommutative in the sense $\omega \wedge \eta = (-1)^{k+l} \omega \wedge \eta$, and associative.

Suppose (x, U) is a chart with standard basis e_i. Let e^i be the dual basis, defined by $e^i(e_j) = \delta^i_j$ for all $p \in U$ and $1 \le i, j \le n$. Then we can locally represent a k-form as

$$\omega = \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} \qquad \forall p \in U.$$

We will also use the shorthand notation $\omega = \sum_I \omega_I e^I$ where $I = (i_1, \ldots, i_k)$ is a multi-index.

Under coordinate transformations, k-forms transform as the change of variables formula states – and therefore their integral over k-dimensional submanifolds will turn out to be well-defined irrespective of a choice of coordinates. This case amounts to $f = \mathrm{id}$ in the following more general statement.

Theorem 34. *Let $F\colon M \to N$ be a differentiable map of n-manifolds, $h \in \mathcal{D}(N)$, and consider the charts (x, U) at $p \in M$ with standard basis e^i, as well as (y, V) at $f(p) \in N$ with standard basis f^i. Then*

$$F^*(hf^1 \wedge \ldots \wedge f^n) = (h \circ F) \det \Big(\frac{\partial (y^i \circ F)}{\partial x^j} \Big)_{ij} e^1 \wedge \ldots \wedge e^n.$$

Define orientation.

Theorem 35. *On a manifold M^n there exists an n-form ω with $\omega(p) \neq 0$ for all $p \in M$ if and only if M is orientable.*

7.3. **The differential.** In order to generalize the fundamental theorem in terms of alternating forms it will be essential to have a notion of differentiability at hand. For the case of 0-forms, differentiation is already defined in terms of the Lie derivative:

$$d\colon \Lambda^0 M = \mathcal{D}(M) \to \Lambda^1 M = \mathcal{V}^*(M) \qquad f \mapsto df, \quad \text{where } df(X) = \partial_X f.$$

Locally, $df = \sum_{j=1}^n \frac{\partial f}{\partial x^j} e^j$. Indeed,

$$\sum_j \frac{\partial f}{\partial x^j} e^j \Big(\sum_i \xi^i e_i \Big) = \sum_j \xi^j \frac{\partial f}{\partial x^j} = df(X).$$

We extend the d-operator to k-forms by applying the differential to the coefficients:

Definition. Let $\omega \in \Lambda^k M$. Then, for each chart (x, U) with standard basis e^i we set

$$d\colon \Lambda^k M \to \Lambda^{k+1} M, \qquad d\omega := \sum_I d\omega_I \wedge e^I = \sum_I \sum_r \frac{\partial \omega_I}{\partial x^r} e^r \wedge e^I. \tag{42}$$

Examples. 1. For the one-form e^i we have $de^i = d(1e_i) = 0$, likewise for the constant coefficient k-forms e^I.
2. If $\omega \in \Lambda^n M$ then $d\omega = 0$.
3a). On $\mathbb{R}^3$ the form $\omega := f_1 e^2 \wedge e^3 + f_2 e^3 \wedge e^1 + f_3 e^1 \wedge e^2$ has $d\omega = \operatorname{div} f e^1 \wedge e^2 \wedge e^3$.

b) For $\eta := g_1 e^1 + g_2 e^2 = g_3 e^3$ on $\mathbb{R}^3$ we have $d\eta = \omega$ with $f = \operatorname{curl} g$. In particular, $d^2\eta = d\omega = \operatorname{div}(\operatorname{curl} g) e^1 \wedge e^2 \wedge e^3 = 0$.

Before showing that $d\omega$ is well-defined, i.e., independent of the coordinate system, we state some properties.

Theorem 36. *d is $\mathcal{D}(M)$-linear and satisfies the product rule*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \qquad \forall \omega \in \Lambda^k M,\ \eta \in \Lambda^l M. \tag{43}$$

Moreover,

$$d^2 = 0, \qquad i.e., \quad d(d\omega) = 0 \quad \forall \omega \in \Lambda^k M.$$

Proof. Linearity is clear. By linearity it is sufficient to check the product rule on $\omega = ge^I$ and $\eta = he^J$. Indeed we have

$$\begin{aligned} d(\omega \wedge \eta) &= d(ge^I \wedge he^J) = d(gh) \wedge e^I \wedge e^J = ((dg)h + gdh) \wedge e^I \wedge e^J \\ &= (dg \wedge e^I) \wedge he^J + (-1)^k ge^I \wedge\ dh \wedge e^J. \end{aligned}$$

Now we prove $d^2 = 0$. By linearity it suffices to consider $\omega = fe^I$. Then

$$d\omega = \sum_{r=1}^n \frac{\partial f}{\partial x^r} e^r \wedge e^I \quad \Rightarrow \quad d(d\omega) = \sum_{r=1}^n \sum_{s=1}^n \frac{\partial^2 f}{\partial x^s \partial x^r} e^s \wedge e^r \wedge e^I,$$

and due to the Schwarz lemma, as well as $e^s \wedge e^r = -e^r \wedge e^s$, the terms of the sum cancel in pairs. □

The equation $d^2 = 0$ captures the Schwarz lemma in a notationally compact way. In fact, many other integrability conditions can be expressed using forms in the most elegant way. For instance, the Frobenius theorem can be formulated as follows: A distribution Δ^k is integrable if and only if the ideal

$$\mathcal{I}(\Delta) := \{\omega \in \Lambda^l(M) : \omega(X_1, \ldots, X_l) = 0 \text{ if } X_1(p), \ldots, X_l(p) \in \Delta^k(p) \forall p \in M\}$$

satisfies $d(\mathcal{I}(\Delta)) \subset \mathcal{I}(\Delta)$. It is a good exercise to check this statement on examples of distributions.

To see that d is well-defined, we state an invariant formula for d:

Theorem 37. *For each k-form ω and vector fields X_i, the form*

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \partial_{X_i}\big(\omega(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1})\big) \\ &\quad + \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\big([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1})\big) \end{aligned} \tag{44}$$

has the local representation (42).

Proof. We first check for $\mathcal{D}(M)$-linearity of (44).

Let us do this only for the case $k = 1$. Then in (44) the first sum contains the two terms $i = 1, 2$, and the second sum only one ($i = 1, j = 2$):

$$d\omega(X,Y) = \partial_X\big(\omega(Y)\big) - \partial_Y\big(\omega(X)\big) - \omega\big([X,Y]\big) \tag{45}$$

This case is the one when (44) is used most frequently.

Then

$$d\omega(fX,Y) - f d\omega(X,Y) = 0 - \partial_Y(f\omega(X)) + f\partial_Y(\omega(X)) - \omega\big([fX,Y] - f[X,Y]\big) = 0,$$

due to $[fX,Y] = f[X,Y] - (\partial_Y f)X$. By anticommutativity of the right hand side of (45) the same holds w.r.t. Y. The general case $k \geq 1$ is only different in that it involves more indices.

We now claim that locally, w.r.t. a chart (x,U), the invariant expression (44) amounts to (42). Using linearity of the two formulas in ω, it is sufficient to consider the case $\omega = fe^I$. By coordinate renumbering we may assume specifically $\omega = fe^1 \wedge \ldots \wedge e^k$. Writing ω as a linear combination of basis multivectors, the $\mathcal{D}(M)$-linear of d implies that it suffices to show the identity for the basis representation of $d\omega$,

$$d\omega = \sum_{1 \leq j_1 < \ldots < j_{k+1} \leq n} d\omega\big(e_{j_1}, \ldots, e_{j_{k+1}}\big) e^{j_1} \wedge \ldots \wedge e^{j_{k+1}}. \tag{46}$$

The second term in (44) vanishes for any of these multivectors, as $[e_i, e_j] = 0$. Consider the i-th term of the first sum,

$$(-1)^{i+1}\partial_{X_i}\big(\omega(e_{j_1}, \ldots, e_{j_{i-1}}, \widehat{e}_{j_i}, e_{j_{i+1}}, \ldots, e_{j_{k+1}})\big). \tag{47}$$

Our particular k-form $\omega = fe^1 \wedge \ldots \wedge e^k$ vanishes indentically on all ordered multivectors except for $(e_1, \ldots, e_k)$. Comparing, we see the cancelled vector $\widehat{e_{j_i}}$ can only occur in the last position, $i = k+1$, and so j_{k+1} can be any index between $k+1$ and n, that is,

$$\big(j_1, \ldots, j_k, j_{k+1}\big) \in \{(1, \ldots, k, r) : k+1 \leq r \leq n\}.$$

Thus only $(-1)^{k+2}\partial_{X_{k+1}}\big(\omega(e_1, \ldots, e_k, e_r)\big)$. contributes to the i-th term, i.e., these are the only indices producing a nonzero contribution in (47).

Consequently, we are left with only one term in (44), namely

$$d\omega\big(e_1, \ldots, e_k, e_r\big) = (-1)^{k+2}\partial_{e_r}\big(\omega(e_1, \ldots, e_k, e_r)\big) = (-1)^k \partial_r f.$$

Hence the representation obtained from (44) is

$$d\omega \overset{(46)}{=} \sum_{r=k+1}^{n} (-1)^k \partial_r f\, e^1 \wedge \ldots \wedge e^k \wedge e^r = \sum_{r=k+1}^{n} \partial_r f\, e^r \wedge e^1 \wedge \ldots \wedge e^k,$$

which coincides with (42). $\square$

`14. Lecture, Thursday 4.2.10`

There is one more important property of d:

Theorem 38. *If $f\colon M \to N$ is differentiable and $\omega \in \Lambda^k N$ then*

$$f^*(d\omega) = d(f^*\omega).$$

In view of the afore-mentioned formulation of the Frobenius theorem this is no surprise.

Note that the notation f^* means two things: First, in the base point of the form, p gets replaced by $f(p)$, second in the multi-vector argument, each X gets replaced by $df X$. Invoking the chain rule, we see the two actions fit together to give $f^*(g^*(\omega)) = (g \circ f)^*(\omega)$ – note the change in order!

Proof. By induction on k. For $k = 0$, we have for $\omega = g \in \mathcal{D}(M)$, as desired,

$$f^*(dg)(X) \overset{\text{def.} f^*}{=} dg\big(df(X)\big) = d(g \circ f)(X) = d(f^*g)(X).$$

The last equality sign comes from the fact that for a 0-form all what f^* does is to replace the base point.

Assuming the formula for $k-1$, it is sufficient to consider $\omega = ge^{i_1} \wedge \ldots \wedge e^{i_k}$: Note first that $de^{i_k} = 0$ and so $d(f^*e^{i_k}) = de^{i_k} \circ df = 0$. Using this at the second equality sign gives

$$\begin{aligned} d(f^*\omega) &= d\Big(\big(f^*g\, e^{i_1} \wedge \ldots \wedge e^{i_{k-1}}\big) \wedge f^*e^{i_k}\Big) \overset{(43)}{=} d\Big(f^*g\, e^{i_1} \wedge \ldots \wedge e^{i_{k-1}}\Big) \wedge f^*e^{i_k} + 0 \\ &\overset{\text{ind. hypoth.}}{=} f^*\Big(d\big(g\, e^{i_1} \wedge \ldots \wedge e^{i_{k-1}}\big)\Big) \wedge f^*e^{i_k} = f^*\Big(dg \wedge e^{i_1} \wedge \ldots \wedge e^{i_{k-1}}\Big) \wedge f^*e^{i_k} \\ &= f^*(d\omega) \end{aligned}$$

□

Remark to myself: Some comments on closed and exact forms, as well as the notation dx, would be appropriate here.

8. Integration of differential forms

8.1. **Integration over cubes.**

Definition. A differentiable function $\sigma\colon [0,1]^k \to M$ is called a *singular k-cube* in a manifold M^n.

Note that differentiability on a non-open set means that there is a differentiable extension to a superset. We do not require that σ is an embedding or an immersion: It may be that $\sigma([0,1]^k)$ has dimension lower than k (in an appropriate sense); however, in that case, the integral defined below will then vanish.

We can define an integral of a k-form over a k-cube as follows:

Definition. *(i)* If ω is an n-form on $[0,1]^n$, then we set

$$\int_{[0,1]^n} \omega := \int_{[0,1]^n} f(x)\, dx.$$

where $\omega(x) = f(x)\, e^1 \wedge \ldots \wedge e^n$ for the basis (e^i) of $\mathbb{R}^{n*}$.
(ii) If ω is a k-form on a manifold M^n, and σ is a singular k-cube in M, we set

$$\int_\sigma \omega := \int_{[0,1]^k} \sigma^*\omega.$$

Example. If $\omega = \sum_i a_i e^i = \langle a, .\rangle$ is a one-form on $\mathbb{R}^n$, and $\sigma\colon [0,1] \to \mathbb{R}^n$ is a 1-cube, then we recover the path integral. Indeed, $d\sigma(e_j) = (\sigma')^j e_j$, and so

$$\int_\sigma \omega \overset{(ii)}{=} \int_{[0,1]} \sigma^*\omega = \int_{[0,1]} \sigma^*\Big(\sum_i a_i e^i\Big) = \int_{[0,1]} \sum_i (a_i \circ \sigma)\, e^i \sigma' = \int_{[0,1]} \langle a\big(\sigma(t)\big), \sigma'(t)\rangle\, dt.$$

Remarks. 1. The most common notation for the dual basis of $\mathbb{R}^k$ is $dx^1 := e^1, \ldots, dx^k := e^k$. Using this notation, *(i)* reads $\int_{[0,1]^k} f\, dx^1 \wedge \ldots \wedge dx^k = \int_{[0,1]^k} f\, dx^1 \ldots dx^k$ and so appears almost tautological. That is, our notation for forms mimics classical notation as much as possible.

2. $\int_\sigma \omega$ counts the image with multiplicity: If σ covers a set twice and with the same orientation (think, for instance, of a doubly covered circle), then each image point contributes twice to the integral.

We want that our definition is parameterization independent, that is, it depends only on the image set $\sigma([0,1]^k)$, but not on the particular parameterization chosen. As in the case of path integrals, this holds for the case of orientation preserving parameter changes:

Proposition 39. *(i) If $\sigma\colon [0,1]^n \to \mathbb{R}^n$ is an injective singular n-cube with* $\det d\sigma \geq 0$ *on $[0,1]^n$, then*

$$\int_\sigma f\, e^1 \wedge \ldots \wedge e^n = \int_{\sigma([0,1]^n)} f(x)\, dx.$$

(ii) Suppose $h\colon [0,1]^k \to [0,1]^k$ is a diffeomorphism with $\det dh \geq 0$*, σ is a singular k-cube in M, and ω is a k-form on M. Then*

$$\int_\sigma \omega = \int_{\sigma \circ h} \omega.$$

Proof. *(i)*

$$\begin{aligned}\int_\sigma f e^N &= \int_{[0,1]^n} \sigma^*(f e^N) \overset{34}{=} \int_{[0,1]^n} (f \circ \sigma) \det d\sigma\, e^N \\ &= \int_{[0,1]^n} (f \circ \sigma) |\det d\sigma|\, dx \overset{\text{ch. of var.}}{=} \int_{\sigma([0,1]^n)} f(x)\, dx\end{aligned}$$

(*ii*)

$$\int_{\sigma\circ h}\omega=\int_{[0,1]^k}(\sigma\circ h)^*\omega=\int_{[0,1]^k}h^*(\sigma^*\omega)\overset{(i)}{=}\int_{[0,1]^k}(\sigma^*\omega)=\int_\sigma\omega$$

$\square$

8.2. **Chains.** Stokes' theorem will involve integration over the boundary of a k-cube. A k-cube has $2k$ bounding faces, and each face can be regarded a $(k-1)$-cube. In order to integrate over the boundary of an k-cube we will simply add up the integrals over all bounding faces. So we need to integrate forms along unions of $(k-1)$-cubes. It is convenient to do this using a more general notion, which represents again an algebraic closure of geometric objects:

Definition. (*i*) A *k-chain* σ in M^n is a linear combination of k-cubes, $\sigma=\sum_{i=1}^l a^i\sigma_i$ where $a^i\in\mathbb{R}$.
(*ii*) We define the integral of a k-form ω over the k-chain σ by

$$\int_{\sum a^i\sigma_i}\omega:=\sum_{i=1}^l\Big(a^i\int_{\sigma_i}\omega\Big).$$

Example. Replacing σ by $-\sigma$ changes the sign of the integral, and so has the same effect as changing the orientation of σ.

Given a k-cube we now want to associate the chain of its boundary $(k-1)$-cubes. We want to denote the $2k$ faces, resulting as boundary restrictions from σ using double indices (i,b), where $1\le i\le k$ and $b=0,1$; the two parallel faces with normal e_i are distinguished by b. So given $\sigma\colon[0,1]^k\to M$ we define for $1\le i\le k$ and $b\in\{0,1\}$

$$\sigma_{i,b}\colon[0,1]^{k-1}\to M,\qquad \sigma_{i,b}(x^1,\dots,x^{k-1}):=\sigma\big(x^1,\dots,x^{i-1},b,x^i,\dots x^{k-1}\big).\tag{48}$$

Examples. 1. In case of a cube $(k=3)$, $\sigma_{3,0}$ parameterizes with the bottom $(z=0)$ face, while $\sigma_{3,1}$ parameterizes with the top $(z=1)$ face.
2. If $k=1$ then $\sigma_{1,0}=\sigma(0)$ while $\sigma_{1,1}=\sigma(1)$

We want to take orientation into account. In case of a square, the desired orientation will be a, say, counterclockwise arrangement of the four edges. In general, one of the parallel faces $\sigma_{i,0}$ and $\sigma_{i,1}$ will be assigned opposite orientations, that is, exactly one of them appears with a minus sign.

Definition. (*i*) The *boundary* of a k-cube σ in M^n with $k\in\mathbb{N}$ is the $(k-1)$-chain

$$\partial\sigma:=\sum_{i=1}^k\sum_{b\in\{0,1\}}(-1)^{i+b}\sigma_{i,b}.$$

For $k = 0$, when $\sigma\colon [0,1]^0 = 0 \to M$, we define $\partial\sigma := 1 \in \mathbb{R}$.
(*ii*) The boundary of a k-chain $\sigma = \sum_{i=1}^l a^i\sigma_i$, where $k \in \mathbb{N}_0$, is

$$\partial\sigma := \sum_{i=1}^l a^i \partial\sigma_i.$$

(*iii*) A chain σ is *closed* if $\partial\sigma = 0$.

Note that the number b makes a pair of parallel boundary faces oppositely oriented.

Examples. 1. Consider a closed curve σ. That is, σ is a 1-cube with $\sigma(1) = \sigma(0)$. Then $\partial\sigma = \sigma_{1,1} - \sigma_{1,0} = \sigma(1) - \sigma(0) = 0$, so that σ is a closed 1-cube.
2. Two curves σ^1, σ^2 with the same endpoints form a 1-chain $\sigma := \sigma^1 - \sigma^2$ which is also closed. Indeed, if $\sigma^1(0) = \sigma^2(0)$ and $\sigma^1(1) = \sigma^2(1)$ then

$$\partial\sigma = \sigma^1_{1,1} - \sigma^1_{1,0} - \sigma^2_{1,1} + \sigma^2_{1,0} = 0$$

3. Consider a 2-cube $\sigma\colon [0,1]^2 \to M$. Then $\partial\sigma = \sigma_{1,1} - \sigma_{2,1} - \sigma_{1,0} + \sigma_{2,0}$. We claim that $\partial\sigma$ is closed. To see this, label the vertices in counterclockwise order as $P = \sigma(0,0)$, $Q = \sigma(1,0)$ etc. Then

$$\partial(\partial\sigma) = (R - Q) - (R - S) - (S - P) + (Q - P) = 0.$$

4. For a 3-cube the 1-chain $\partial(\partial\sigma)$ is a sum over $24 = 6 \cdot 4$ edges of the cube. Due to the sign convention, pairs of edges cancel, so that again $\partial(\partial\sigma) = 0$.

The property $\partial(\partial\sigma) = 0$, displayed in the examples, is analogous to $d^2 = 0$ for forms, and holds in general:

Proposition 40. *If σ is a k-chain in M^n, then*

$$\partial^2\sigma := \partial(\partial\sigma) = 0.$$

Proof. It is sufficient to check this for k-cubes. This can be done by a longer calculation. □

Remark. The only interesting feature in the definition of ∂ is the sign. In view of the proposition we can say that the exponent $i + b$ is chosen such that it assigns a pair of adjacent faces of a k-cube an opposite orientation, in the sense that taking once again the boundary of the $(k-1)$-dimensional faces yields the opposite orientation of the $(k-2)$-dimensional faces.

Remarks. 1. While k-cubes are well adapted to coordinate parallel integration as needed for Stokes theorem, there is another setting more widely used in algebraic topology: *k-simplices* are build from triangles rather than squares, and have a similar boundary operator ∂. See [W], for instance, for this approach.
2. More generally, a class of objects indexed with $k \in \mathbb{N}_0$ and an operator d going from

the $(k+1)$-objects to the k-objects (or vice versa) such that $d^2 = 0$ form a *chain complex.* On these complexes, a homology theory can be defined.

8.3. **Stokes' theorem for chains.** We can now prove a simple case of Stokes' theorem:

Theorem 41. *If σ is a k-chain in a manifold M^n and ω is a $(k-1)$-form on M then*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Proof. 1. Consider the case $M = \mathbb{R}^n$, $k = n$, and $\sigma = \mathrm{id}$. By linearity it is sufficient to prove the theorem for the particular $(n-1)$-form

$$\omega = f e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n.$$

Then

$$d\omega = \partial_i f\, e^i \wedge e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n = (-1)^{i-1} \partial_i f\, e^1 \wedge \ldots \wedge e^n.$$

Using Fubini's theorem as well as the fundamental theorem of calculus applied to integration w.r.t. the i-th variable, gives

$$\begin{aligned} \int_{[0,1]^n} d\omega &= \int_{[0,1]^n} (-1)^{i-1} \partial_i f(x^1, \ldots, x^i, \ldots, x^n)\, dx^1 \ldots dx^i \ldots dx^n \\ &= (-1)^{i-1} \int_{[0,1]^{n-1}} f(x^1, \ldots, 1, \ldots, x^n) - f(x^1, \ldots, 0, \ldots, x^n)\, dx^1 \ldots \widehat{dx^i} \ldots dx^n. \end{aligned} \tag{49}$$

On the other hand, let us introduce the notation to parameterize the j-th pair of bounding faces,

$$\iota^n_{j,b} \colon [0,1]^{n-1} \to [0,1]^n, \qquad \iota^n_{j,b}(x^1, \ldots, x^{n-1}) := \big(x^1, \ldots, x^{j-1}, b, x^j, \ldots x^{n-1}\big), \tag{50}$$

compare with (48). Then using the shorthand notation id^n for $\mathrm{id}|_{[0,1]^n}$ we can write

$$\int_{\partial\,\mathrm{id}^n} \omega = \sum_{j=1}^n \sum_{b\in\{0,1\}} (-1)^{j+b} \int_{\iota^n_{j,b}} \omega = \sum_{j=1}^n \sum_{b\in\{0,1\}} (-1)^{j+b} \int_{[0,1]^{n-1}} (\iota^n_{j,b})^* f e^1 \wedge \ldots \wedge \widehat{e^j} \wedge \ldots \wedge e^n.$$

We claim that only the two terms with $j = i$ contribute. To see this, note that the form will vanish on multivectors which come from coordinate-parallel hyperplanes not perpendicular to e_i. This can be seen by direct calculation: The differential of (50) is

$$d(\iota^n_{j,b})(e_1) = e_1, \ldots, d(\iota^n_{j,b})(e_{j-1}) = e_{j-1}, \;\; d(\iota^n_{j,b})(e_j) = e_{j+1}, \ldots, d(\iota^n_{j,b})(e_{n-1}) = e_n$$

and hence ω vanishes on the linear hull of these fields except for the case $i = j$. We conclude

$$\int_{\partial\,\mathrm{id}^n} \omega = \sum_{b\in\{0,1\}} (-1)^{i+b} \int_{[0,1]^{n-1}} f\big(\iota^n_{i,b}(x)\big)\, dx^1 \ldots \widehat{dx^i} \ldots dx^n,$$

which agrees with (49). Thus we have established Stokes' theorem for our special case.

2. For σ a k-cube in M^n, step 1 gives

$$\int_\sigma d\omega \overset{\text{def.}}{=} \int_{\mathrm{id}^k} \sigma^*(d\omega) \overset{(38)}{=} \int_{\mathrm{id}^k} d(\sigma^*\omega) \overset{(i)}{=} \int_{\partial\,\mathrm{id}^k} \sigma^*\omega = \int_{\partial\sigma} \omega.$$

3. The generalization to k-chains is immediate. □

Note that the manifold dimension n can be larger than k, but the integral will not see the extra dimensions.

`15. Lecture, Thursday 11.2.10` ______

Examples. 1. In case $k = 1$ we can apply the theorem to a 0-form f, and the 1-chain id^1. Then $\int_{\mathrm{id}^1} df = \int_{\partial\,\mathrm{id}^1} f$ which means $\int_{[0,1]} f'(x)\,dx = f(1) - f(0)$.
2. If we take an alternating sum over the forms used in step 1 of the proof we obtain the divergence theorem. To see this, consider the $(n-1)$-form

$$\omega = \sum_i (-1)^{i+1} f_i\, e^1 \wedge \ldots \wedge \widehat{e^i} \wedge \ldots \wedge e^n \quad \text{with} \quad d\omega = \operatorname{div} f\, e^1 \wedge \ldots \wedge e^n$$

on a standard cube. Writing $\omega = \sum_i \omega_i$ we see that only the two i-faces $\iota^n_{i,b}$ of $\partial[0,1]^n$ contribute to $\int_{\partial[0,1]^n} \omega_i$. Hence the left hand side of Stokes' theorem reads

$$\int_{\partial[0,1]^n} \omega = \sum_{i,b} (-1)^{i+b} \int_{\iota^n_{i,b}} \omega_i = \sum_{i,b} (-1)^{b+1} \int_{[0,1]^{n-1}} f_i\big(\iota_{i,b}(x)\big)\,dx = \sum_{i,b} \int_{[0,1]^{n-1}} \langle f \circ \iota_{i,b}, \nu_i\rangle\,dx,$$

where ν_i is the exterior normal to the face $\iota_{i,b}$. Let ν be the exterior normal to $\partial[0,1]^n$, which is defined except on a set of measure 0, namely the $(k-2)$-dimensional faces of the cube. Then we can write $\int_{\partial[0,1]^n} \langle f, \nu\rangle\,dx_{n-1}$ for the right hand side. Hence for our case, Stokes' theorem for forms $\int_{[0,1]^n} \operatorname{div} f\, e^N = \int_{\partial[0,1]^n} \omega$ gives the classical divergence theorem, written with surface integrals: $\int_{[0,1]^n} \operatorname{div} f\,dx = \int_{\partial[0,1]^{n-1}} \langle f, \nu\rangle\,dx_{n-1}$. It is worthwhile to derive the same formula for any immersed singular n-cube σ.

8.4. **Integration of forms over manifolds.** Let us first define orientation:

Definition. (i) Two charts (x, U) and (y, V) of a manifold are *orientation compatible* if the transition map satsifies

$$\det d(y \circ x^{-1}) > 0 \quad \text{for all } p \in x(U \cap V). \tag{51}$$

(ii) An *orientation* of a manifold M is an atlas $\mathcal{A} = \{(x_\alpha, U_\alpha) : \alpha \in A\}$ whose charts are pairwise orientation compatible.
(iii) A manifold M is *orientable* if it has an orientation.

Example. Möbius band and Klein bottle are non-orientable 2-manifolds, $\mathbb{R}P^n$ is a non-orientable n-manifold.

The following is not hard to check:
• $\det d(y \circ x^{-1}) > 0$ implies $\det d(x \circ y^{-1}) > 0$.

• If two charts with a connected nonempty intersection set are not orientation preserving then composing one chart with an orientation reversing diffeomorphism of $\mathbb{R}^n$ makes them orientation compatible.

• Suppose $(M, \mathcal{A})$ is orientable and a one more chart is given which is orientation compatible with one chart of $\mathcal{A}$ on a nonempty intersection set. Then this chart will be orientation compatible with all charts of $\mathcal{A}$. To prove this, use that the left hand side of (51) is continuous and that M is connected by our general assumptions.

• Thus on an orientable manifold, there are exactly two differentiable structures $(M, \mathcal{S}_+)$, $(M, \mathcal{S}_-)$ which give an orientation, sometimes called positive and negative orientation, or direct and indirect orientation. If we were to admit non-connected manifolds with $k \in \mathbb{N}$ connected components, then we had 2^k orientations.

On an oriented manifold $(M, \mathcal{A})$, we call an n-tuple of linearly independent tangent vectors $(v_1, \dots, v_n) \in (T_pM)^n$ *(positively) oriented* if the orientation of their principal parts,

$$\mu(v_1, \dots, v_n) := \operatorname{sign}\big(\det(dx(v_1), \dots, dx(v_n))\big) \tag{52}$$

is $+1$ and not -1 for $x \in \mathcal{A}$.

Given an oriented manifold $(M^n, \mathcal{A})$, we call a local diffeomorphism $\sigma\colon \Omega \subset \mathbb{R}^n \to M$ *orientation preserving* if σ^{-1} is orientation compatible with $(M, \mathcal{A})$, and orientation reversing if it is not. We can now formulate a crucial property, the parametrization invariance of the integral of forms over oriented chains:

Lemma 42. *Let M^n be an oriented manifold, and σ_1, σ_2 be singular n-chains which are the sum of orientation preserving diffeomorphic n-cubes. Then*

$$\operatorname{supp} \omega \subset \sigma_1([0,1]^n) \cap \sigma_2([0,1]^n) \qquad \textit{implies} \qquad \int_{\sigma_1} \omega = \int_{\sigma_2} \omega.$$

Proof. In $\sigma_2^{-1}(\operatorname{supp} \omega)$ we can write $\sigma_2 = \sigma_1 \circ (\sigma_1^{-1} \circ \sigma_2)$ and so σ_2 is a reparameterization of σ_1, preserving orientation. Hence the result follows from the proof of the parameterization invariance Prop. 39(ii). □

Let us emphasize what is hidden in the proof: the lemma holds since the forms and multiple integrals obey the very same transformation rule.

Consequently, $\int_M \omega := \int_\sigma \omega$ is well-defined for all diffeomorphic n-cubes, compatible with the orientation of M, such that $\operatorname{supp} \omega \subset \sigma([0,1]^n)$.

Consider an oriented manifold $(M^n, \mathcal{A})$. By modifying charts suitably, one can show there is a countable open covering $\{U_\alpha : \alpha \in \mathbb{N}\}$ of M with charts (x_α, U_α) such that each $\sigma_\alpha := x_\alpha^{-1}$ is a orientation preserving diffeomorphic n-cube. Indeed, we could cover $x_\alpha(U)$ with coordinate parallel open cubes in a locally finite way. The restriction of the x_α to these cubes then defines charts which cover, i.e. define an atlas.

Let now φ_α be a *partition of unity* subordinate to $\{U_\alpha\}$, that is, a family of functions $\{\varphi_\alpha\colon U_\alpha \to [0,1] : \alpha \in \mathbb{N}\}$ such that
• $\operatorname{supp} \varphi_\alpha \subset\subset U_\alpha$ and
• $\sum_{\alpha\in\mathbb{N}} \varphi_\alpha \equiv 1$ on M.
The existence of such a partition can be shown to follow from second countability (see [Lee]).

For ω an n-form on M^n we then define

$$\int_M \omega := \sum_{\alpha\in\mathbb{N}} \int_M \varphi_\alpha \omega.$$

This definition is independent of the covering, as

$$\sum_{\alpha\in\mathbb{N}} \int_M \varphi_\alpha\omega = \sum_{\alpha\in\mathbb{N}} \int_M \Big(\sum_{\beta\in\mathbb{N}} \psi_\beta\Big)\varphi_\alpha\omega = \sum_{\alpha,\beta\in\mathbb{N}} \int_M \psi_\beta\varphi_\alpha\omega = \sum_{\beta\in\mathbb{N}} \int_M \Big(\sum_{\alpha\in\mathbb{N}} \varphi_\alpha\Big)\psi_\beta\omega = \sum_{\beta\in\mathbb{N}} \int_M \psi_\beta\omega.$$

Note that the orientation is implicit in our definition and a change of orientation will result in a sign change. Thus, if we were to employ a more precise notation, we would write

$$\int_{(M,\mathcal{S}_+)} \omega = -\int_{(M,\mathcal{S}_-)} \omega.$$

8.5. **Manifolds with boundary.** Let us define the upper half of a ball by

$$\overline{B}^n_+ := \{x \in \mathbb{R}^n : \|x\| < 1 \text{ and } x^n \geq 0\}.$$

Note that the bounding $(n-1)$-ball in the plane $\{x^n = 0\}$ is included. We now extend our notion of manifolds to allow for boundary:

Definition. (i) A *topological manifold with boundary* of dimension $n \in \mathbb{N}$ is a topological space M which is Hausdorff, second countable, and such that each point has a neighbourhood homeomorphic to either B^n or $\overline{B}^n_+$. If M has an atlas of charts with differentiable transition maps then M is a *(differentiable) manifold with boundary.*
(ii) The *boundary* ∂M of a manifold M with boundary is the set of those points $p \in M$ which do not have a neighbourhood homeomorphic to B^n.

The notion of a boundary in (ii) is well-defined since there is no homeomorphism (or diffeomorphism) from $\overline{B}^n_+$ onto B^n. Also, each connected component of ∂M is a manifold of its own, whose charts are given by the restriction of the charts $x\colon U \to \overline{B}^n_+$ to $x^{-1}\big(\overline{B}^n_+ \cap \{x^n = 0\}\big)$.

By definition, a manifold in the usual sense can also be considered a manifold with (empty) boundary. It is common to say *closed manifold* to emphasize that a manifold has no boundary, $\partial M = \emptyset$ (nevertheless, considered as a topological space, a manifold with boundary is also closed).

Examples. 1. $\{x \in \mathbb{R}^n : x_n \geq 0\}$
2. $\mathbb{S}^n \cap \{x : x^n \geq 0\}$.
3. If $\psi\colon \mathbb{R}^n \to \mathbb{R}$ with $\operatorname{grad}\psi \neq 0$ on $\psi^{-1}(0)$ then the implicit function theorem gives that $M = \psi^{-1}([0,\infty))$ is a differentiable manifold with boundary $\partial M = \psi^{-1}(0)$.

In order to have $\int_{\partial M}\omega$ well-defined, where ω is an $(n-1)$-form, we need to define an orientation of ∂M. The idea is simple to explain for a submanifold like $\mathbb{S}^{n-1} = \partial B^n \subset \mathbb{R}^n$: Here we define tangent vectors $(v_1, \ldots, v_{n-1}) \in T_p\mathbb{S}^{n-1}$ to be positively oriented for $\mathbb{S}^{n-1}$ if with respect to the exterior normal $\nu(p) = p$ to ∂B^n at p the n vectors $(p, v_1, \ldots, v_{n-1})$ are positively oriented in $\mathbb{R}^n$.

At a point $p \in \partial M$, let us call a tangent vector $v \in T_pM$ *outward pointing* if its principal part w.r.t. to a chart $x\colon U \to \overline{B}^n_+$ is negative, $dx^n(v) = \xi^n < 0$. In particular, an outward pointing vector cannot be linearly dependent on any tangent vectors to ∂M, since the latter satisfy $\xi^n = 0$.

Definition. Let M^n be oriented, and $v_1, \ldots, v_{n-1}$ linearly independent tangent vectors to ∂M at $p \in \partial M$. If v is an outward pointing vector at p we define the *induced orientation* of ∂M by

$$\mu_{\partial M}(v_1, \ldots, v_{n-1}) := \mu_M(v, v_1, \ldots, v_{n-1}).$$

Example. For the upper half-space $M := \mathbb{R}^n \cap \{x^n \geq 0\}$ the vector $-e_n$ is an outward pointing normal. Then

$$\mu_{\partial M}(e_1, \ldots, e_{n-1}) \overset{\text{def}}{=} \mu_M(-e_n, e_1, \ldots, e_{n-1}) = (-1)^n \mu_M(e_1, \ldots, e_n) \tag{53}$$

and so for n odd the orientation of $\partial M = \mathbb{R}^{n-1} \times \{0\}$ differs from the standard orientation of $\mathbb{R}^{n-1}$.

Suppose σ is an orientation preserving (diffeomorphic) singular n-cube in an oriented manifold M, such that its n-th bottom face parameterizes a subset of ∂M, that is,

$$\partial M \cap \sigma\big([0,1]^n\big) = \sigma_{n,0}\big([0,1]^{n-1}\big). \tag{54}$$

By (53),

$$\sigma_{n,0}\colon [0,1]^{n-1} \to (\partial M, \text{induced orientation})$$

is orientation preserving for even n, and reversing for odd n. Consequently, if ω is an $(n-1)$-form on M with compact support in $\sigma\big((0,1)^n \cup (\{0\} \times (0,1)^{n-1})\big)$ we have

$$\int_{\sigma_{n,0}} \omega = (-1)^n \int_{\partial M} \omega.$$

On the other hand, $\sigma_{n,0}$ appears with coefficient $(-1)^n$ in $\partial\sigma$ and so

$$\int_{\partial\sigma} \omega = \int_{(-1)^n \sigma_{n,0}} \omega = (-1)^n \int_{\sigma_{n,0}} \omega = \int_{\partial M} \omega. \tag{55}$$

Note that the outer sides of the equation are equal without any extra signs. This means that the definition of induced orientation and the sign in the definition of the boundary operator for chains are consistent for our purposes.

8.6. **Stokes' theorem for manifolds.**

Theorem 43. *If M is an oriented n-dimensional manifold with boundary ∂M (given the induced orientation), and ω is an $(n-1)$-form with compact support then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

The assumption that ω has compact support is needed to guarantee that the integrals exist, just as $\int_{\mathbb{R}} f' = 0$ is valid for f with compact support. In case M itself is compact, the condition is superfluous.

Proof. The manifold M has a countable open cover $\mathcal{O} = \{U_\alpha \subset M \text{ open} : \alpha \in A\}$ with the images of singular orientation preserving diffeomorphic n-cubes, and which are either interior, or parameterize the boundary with their n-th bottom face as in (54). We let $\{\varphi_\alpha : \alpha \in \mathbb{N}\}$ be a partition of unity subordinate to $\mathcal{O}$; as in the previous subsection we require for boundary cubes that the support of φ_α is in $\sigma\big((0,1)^n \cup (\{0\} \times (0,1)^{n-1})\big)$. Finitely many indices suffice to cover the compact set $\operatorname{supp}\omega$.

For each $\alpha \in \mathbb{N}$ such that U_α is a an interior n-cube, Stokes' theorem for chains gives that

$$\int_{U_\alpha} d(\varphi_\alpha \omega) \overset{\text{Thm. 41}}{=} \int_{\partial U_\alpha} \varphi_\alpha \omega; \tag{56}$$

in fact, the right hand side vanishes, due to compact support of $\varphi_\alpha\omega$. On the other hand, for each $\alpha \in N$ such that U_α is the image of a boundary cube

$$\int_{U_\alpha} d(\varphi_\alpha \omega) = \int_{\partial U_\alpha} \varphi_\alpha \omega. \overset{(55)}{=} \int_{\partial M} \varphi_\alpha \omega, \tag{57}$$

With sums which are finite at every point we have

$$\sum_{\alpha\in\mathbb{N}} d\varphi_\alpha = d\sum_{\alpha\in\mathbb{N}} \varphi_\alpha = d\,1 = 0 \quad \Rightarrow \quad \sum_{\alpha\in\mathbb{N}} d\varphi_\alpha \wedge \omega = 0 \quad \Rightarrow \quad \sum_{\alpha\in\mathbb{N}} \int_M d\varphi_\alpha \wedge \omega = 0.$$

Thus we can sum (56) and (57) to obtain

$$\int_M d\omega \overset{\text{def.}}{=} \sum_{\alpha\in\mathbb{N}} \int_{U_\alpha} \varphi_\alpha \, d\omega = \sum_{\alpha\in\mathbb{N}} \int_{U_\alpha} d\varphi_\alpha \wedge \omega + \varphi_\alpha d\omega = \sum_{\alpha\in\mathbb{N}} \int_{U_\alpha} d(\varphi_\alpha\omega)$$
$$\overset{(56),(57)}{=} \sum_{\alpha\in\mathbb{N}} \int_{\partial M} \varphi_\alpha\omega \overset{\text{def.}}{=} \int_{\partial M} \omega.$$

□

Corollary 44. *If M is orientable, compact and without boundary then $\int_M d\omega = 0$ for all $\omega \in \Lambda^{n-1}M$.*

There are various topics which could now be addressed:
1. The recovery of all classical integral formulas from Stokes theorem: divergence theorem, Green's theorem, the classical Stokes theorem.
2. Closed and exact forms: A form $\eta \in \Lambda^k M$ is called *exact* if $\eta = d\omega$ for some $\omega \in \Lambda^{k-1}M$. The so-called Poincaré-Lemma says that on a contractible domain, for instance on $\mathbb{R}^n$, a closed form is exact. On the other hand, since a compact orientable manifold has a volume form η with $\int_M \eta > 0$, the above corollary shows that compact manifolds are not contractible.
3. The de Rham cohomology groups

$$H^k(M) := \frac{\{\text{closed forms in } \Lambda^k M\}}{\{\text{exact forms in } \Lambda^k M\}}$$

capture the topology of M. A further topic in this context is the mapping degree, in particular a proof of the Brouwer fixed point theorem.

8.7. **Problems.**

Problem 28 – SKEW SYMMETRIC BILINEAR FORMS:

The purpose of this problem is to prepare the class on multilinear algebra.

a) Give an example of a skew-symmetric bilinear form, $b\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, that is, $b(v,w) = -b(w,v)$ for all $v, w \in \mathbb{R}^n$.
b) Show that $b(v,v) = 0$ for all $v \in \mathbb{R}^n$ is equivalent to b skew-symmetric.
c) What is the dimension of the space of skew-symmetric forms $B(n)$? Exhibit a basis for $B(n)$, for instance in terms of the basis $e^i = \langle ., e_i \rangle$ of the dual space.
d) Can you find a projection which maps an arbitrary bilinear form to the skew-symmetric forms? What are the reasonable properties to ask for?

Problem 29 – QUIZ:

a) For given $f \in \mathcal{V}(\mathbb{R}^n)$ find a form ω on $\mathbb{R}^n$ such that $d\omega = \operatorname{div} f e^1 \wedge \ldots \wedge e^n$.

b) Determine $\{v \in \mathbb{R}^2 : e^1 \wedge e^2(v, e_2) = 0\}$

c) For $w \in \mathbb{R}^3$ given determine $V(w) := \{v \in \mathbb{R}^3 : e^1 \wedge e^2(v, w) = 0\}$

d) Determine $L := \{\omega \in \Lambda^2\mathbb{R}^n : \omega(e_1, e_2) = 0\}$.

e) Let $\omega \in \Lambda^k M$ and $X_1, \dots, X_k \in \mathcal{V}(M)$. Which of the following statements is true?
• The value of $d\omega(X_1, \dots, X_k)$ at $p \in M$ depends only on the values of the X_i's at p, but not on the way they extend to M,
• this value depends only on the value of ω at p but not of the way the form ω extends to M.

Problem 30 – n-DIMENSIONAL CUBE:

Denote the standard unit cube by $C := \{x \in \mathbb{R}^n : 0 \le x^1, \dots, x^n \le 1\}$.

a) Write down the faces of the standard cube (how many are there?).

b) For $1 \le i \le n$ let
$$\omega_i \in \Lambda^{n-1}\mathbb{R}^n, \qquad \omega_i := e^1 \wedge \dots \wedge \widehat{e^i} \wedge \dots \wedge e^n.$$
Describe those faces of C such that the form ω_i vanishes on multivectors formed by tangent vectors to the faces.

Problem 31 – GEOMETRIC INTERPRETATION OF A TWO-FORM:

Let $P(v, w)$ be the planar parallelogram in $\mathbb{R}^3$, spanned by $v, w \in \mathbb{R}^3$. Let $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ be projection to the xy-plane and let $\eta = e^1 \wedge e^2 \in \Lambda^2\mathbb{R}^3$.

a) Give a formula for the signed area of $\pi(P(v, w))$.

b) Prove that $\eta(v, w)$ agrees with the signed area of $\pi(P(v, w))$.

Problem 32 – DECOMPOSABLE AND INDECOMPOSABLE FORMS:

a) Show that in $\mathbb{R}^3$ any two-forms $\omega := v \wedge w$ and $\eta := r \wedge s$ have a sum $\omega + \eta = a \wedge b$ for some covectors $a, b \in \mathbb{R}^{3*}$.

b) Prove that $e^1 \wedge e^2 + e^3 \wedge e^4 \in \Lambda^2\mathbb{R}^4$ cannot be written in the form $v \wedge w$ for $v, w \in \mathbb{R}^{4*}$.

c) Find $\omega \in \Lambda^2\mathbb{R}^4$ such that $\omega \wedge \omega \ne 0$.

Problem 33 – HODGE STAR:

Let V^n be a vector space with inner product. With resprect to an orthonormal basis $(e^1, \dots, e^n)$, define an operator
$$*\colon \Lambda^k V \to \Lambda^{n-k} V, \qquad *(e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}) = e_{i_{k+1}} \wedge e_{i_{k+2}} \wedge \dots \wedge e_{i_n},$$
if $\{i_1, \cdots i_k, i_{k+1} \cdots i_n\}$ is an even permutation of $\{1, 2, \dots, n\}$.

a) What is $*(e^1 \wedge e^2)$ in $\mathbb{R}^3$? What is $*1$ in $\mathbb{R}^n$?

b) Prove $** = (-1)^{k(n-k)}$.

c) Prove that on $\Lambda^k V$ we can define an inner product by

$$\langle v, w \rangle := *(w \wedge *v).$$

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