Appendix C
Measure and Integration Theory

We begin with some general set-theoretic notions. Let Ω be a set. Then its **power set** is denoted by

$$\mathscr{P}(\Omega) := \{A : A \subset \Omega\}.$$

Given $A \subset \Omega$ its **complement** is denoted by $A^{\mathrm{c}} := \Omega \setminus A$, and its **characteristic function** $\mathbf{1}_A$ is defined by

$$\mathbf{1}_A(x) := \begin{cases} 1 & (x \in A) \\ 0 & (x \notin A) \end{cases}$$

for $x \in \Omega$. One often writes $\mathbf{1}$ in place of $\mathbf{1}_\Omega$ if the reference set Ω is understood. For a sequence $(A_n)_n \subset \mathscr{P}(\Omega)$ we write $A_n \searrow A$ if

$$A_n \supset A_{n+1} \quad (n \in \mathbb{N}) \quad \text{and} \quad \bigcap\nolimits_{n\in\mathbb{N}} A_n = A.$$

Similarly, $A_n \nearrow A$ is short for

$$A_n \subset A_{n+1} \quad (n \in \mathbb{N}) \quad \text{and} \quad \bigcup\nolimits_{n\in\mathbb{N}} A_n = A.$$

A family $(A_\iota)_\iota \subset \mathscr{P}(\Omega)$ is called **pairwise disjoint** if $\iota \neq \eta$ implies that $A_\iota \cap A_\eta = \emptyset$. A subset $\mathcal{E} \subset \mathscr{P}(\Omega)$ is often called a **set system** on Ω. A set system is called $\cap$-stable ($\cup$-stable, $\setminus$-stable) if $A, B \in \mathcal{E}$ implies that $A \cap B$ ($A \cup B$, $A \setminus B$) belongs to $\mathcal{E}$ as well. If $\mathcal{E}$ is a set system, then any mapping $\mu : \mathcal{E} \longrightarrow [0, \infty]$ is called a (positive) **set function**. Such a set function is called **σ-additive** if

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

whenever $(A_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ is pairwise disjoint and $\bigcup_n A_n \in \mathcal{E}$. Here we adopt the convention that

$$a + \infty = \infty + a = \infty \qquad (-\infty < a \leq \infty).$$

A similar rule holds for sums $a+(-\infty)$ where $a \in [-\infty, \infty)$. The sum $\infty + (-\infty)$ is not defined. Other conventions for computations with the values $\pm\infty$ are:

$$0 \cdot \pm\infty = \pm\infty \cdot 0, \quad \alpha \cdot \pm\infty = \pm\infty \cdot \alpha = \pm\infty \quad \beta \cdot \pm\infty = \pm\infty \cdot \beta = \mp\infty$$

for $\beta < 0 < \alpha$. If $f : \Omega \longrightarrow \Omega'$ is a mapping and $B \subset \Omega'$ then we denote

$$[f \in B] := f^{-1}(B) := \{x \in \Omega \, : \, f(x) \in B\}.$$

Likewise, if $P(x_1, \dots x_n)$ is a property of n-tuples $(x_1, \dots, x_n) \in (\Omega')^n$ and $f_1, \dots f_n : \Omega \longrightarrow \Omega'$ are mappings, then we write

$$[P(f_1, \dots f_n)] := \{x \in \Omega : P(f_1(x), \dots, f_n(x)) \text{ holds}\}.$$

E.g., for $f, g : \Omega \longrightarrow \Omega'$ we abbreviate $[f = g] := \{x \in \Omega \, : \, f(x) = g(x)\}$.

C.1 σ-Algebras

Let Ω be any set. A **σ-algebra** is a collection $\Sigma \subset \mathscr{P}(\Omega)$ of subsets of Ω, such that the following hold:

1) $\emptyset, \Omega \in \Sigma$.
2) If $A, B \in \Sigma$ then $A \cup B, A \cap B, A \setminus B \in \Sigma$.
3) If $(A_n)_{n\in\mathbb{N}} \subset \Sigma$, then $\bigcup_{n\in\mathbb{N}} A_n$, $\bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

If a set system Σ satisfies merely 1) and 2), it is called an **algebra**, and if Σ satisfies just 2) and $\emptyset \in \Sigma$, then it is called a **ring**. A pair (Ω, Σ) with Σ being a σ-algebra on Ω is called a **measurable space**.

An arbitrary intersection of σ-algebras over the same set Ω is again a σ-algebra. Hence for $\mathcal{E} \subset \mathscr{P}(\Omega)$ one can form

$$\sigma(\mathcal{E}) := \bigcap \{\Sigma \, : \, \mathcal{E} \subset \Sigma \subset \mathscr{P}(\Omega), \, \Sigma \text{ a } \sigma\text{-algebra}\},$$

the σ-algebra **generated** by $\mathcal{E}$. It is the smallest σ-algebra that contains all sets from $\mathcal{E}$. If $\Sigma = \sigma(\mathcal{E})$, we call $\mathcal{E}$ a **generator** of Σ.

If Ω is a topological space, the σ-algebra generated by all open sets is called the **Borel σ-algebra** $\mathfrak{B}(\Omega)$. By 1) and 2), $\mathfrak{B}(\Omega)$ contains all closed sets as well. A set belonging to $\mathfrak{B}(\Omega)$ is called a **Borel set** or **Borel measurable**.

Lemma C.1. *Let Ω be a topological space, and let $A \subset \Omega$ with the subspace topology. Then $\mathfrak{B}(A) = \{A \cap B \, : \, B \in \mathfrak{B}(\Omega)\}$.*

Consider the example that $\Omega = [-\infty, \infty]$ is the **extended real line**. This becomes a compact metric space via the (order-preserving) homeomorphism

$$\arctan : [-\infty, \infty] \longrightarrow [-\pi/2, \pi/2].$$

The subspace topology of $\mathbb{R}$ coincides with its natural topology. The Borel algebra $\mathfrak{B}([-\infty,\infty])$ is generated by $\{(\alpha,\infty] : \alpha \in \mathbb{R}\}$.

A **Dynkin system** (also called **λ-system**) on a set Ω is a subset $\mathcal{D} \subset \mathscr{P}(\Omega)$ with the following properties:

1) $\Omega \in \mathcal{D}$.
2) If $A, B \in \mathcal{D}$ and $A \subset B$ then $B \setminus A \in \mathcal{D}$.
3) If $(A_n)_n \subset \mathcal{D}$ then $\bigcup_n A_n \in \mathcal{D}$.

Theorem C.2 (Dynkin). *If $\mathcal{D}$ is a Dynkin system and $\mathcal{E} \subset \mathcal{D}$ is $\cap$-stable, then $\sigma(\mathcal{E}) \subset \mathcal{D}$.*

The proof is in [Bauer (1990), p.8] and [Billingsley (1979), Thm. 3.2].

C.2 Measures

Let Ω be a set and $\Sigma \subset \mathscr{P}(\Omega)$ a σ-algebra of subsets of Ω. A (positive) **measure** is a σ-additive set function

$$\mu : \Sigma \longrightarrow [0, \infty].$$

In this case the triple (Ω, Σ, μ) is called a **measure space** and the sets in Σ are called **measurable sets**. If $\mu(\Omega) < \infty$, the measure is called **finite**. If $\mu(\Omega) = 1$, it is called a **probability measure** and (Ω, Σ, μ) is called a **probability space**. Suppose $\mathcal{E} \subset \Sigma$ is given and there is a sequence $(A_n)_n \subset \mathcal{E}$ such that

$$\mu(A_n) < \infty \quad (n \in \mathbb{N}) \quad \text{and} \quad \Omega = \bigcup_{n \in \mathbb{N}} A_n;$$

then the measure μ is called **σ-finite with respect to** $\mathcal{E}$. If $\mathcal{E} = \Sigma$, we simply call it σ-finite.

From the σ-additivity of the measure one derives the following properties:

a) **(Finite Additivity)** $\mu(\emptyset) = 0$ and

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) \qquad (A, B \in \Sigma).$$

b) **(Monotonicity)** $A, B \in \Sigma, \quad A \subset B \implies \mu(A) \leq \mu(B)$.
c) **(σ-Subadditivity)** $(A_n)_n \subset \Sigma \implies \mu(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.

See [Billingsley (1979), p.134] for the elementary proofs.

An application of Dynkin's theorem yields the uniqueness theorem.

Theorem C.3 (Uniqueness Theorem). [Billingsley (1979), Thm. 10.3]
Let $\Sigma = \sigma(\mathcal{E})$ with $\mathcal{E}$ being $\cap$-stable. Let μ, ν be two measures on Σ, both σ-finite with respect to $\mathcal{E}$. If μ and ν coincide on $\mathcal{E}$, they are equal.

C.3 Construction of Measures

An **outer measure** on a set Ω is a mapping

$$\mu^* : \mathscr{P}(\Omega) \longrightarrow [0, \infty]$$

such that $\mu^*(\emptyset) = 0$ and μ^* is monotone and σ-subadditive.

Theorem C.4 (Carathéodory). [Billingsley (1979), Thm. 11.1] *Let μ^* be an outer measure on the set Ω. Define*

$$\mathscr{M}(\mu^*) := \big\{ E \subset \Omega : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) \quad \forall A \subset \Omega \big\}.$$

Then $\mathscr{M}(\mu^*)$ is a σ-algebra and $\mu^*\big|_{\mathscr{M}(\mu^*)}$ is a measure on it.

The set system $\mathcal{E} \subset \mathscr{P}(\Omega)$ is called a **semi-ring** if it satisfies the following two conditions:

1) $\mathcal{E}$ is $\cap$-stable and $\emptyset \in \mathcal{E}$.
2) If $A, B \in \mathcal{E}$ then $A \setminus B$ is a disjoint union of members of $\mathcal{E}$.

An example of such a system is $\mathcal{E} = \{(a, b] : a \leq b\} \subset \mathscr{P}(\mathbb{R})$. If $\mathcal{E}$ is a semi-ring then the system of all disjoint unions of members of $\mathcal{E}$ is a ring.

Theorem C.5 (Hahn). [Billingsley (1979), p.140] *Let $\mathcal{E}$ be a semi-ring on a set Ω and let $\mu : \mathcal{E} \longrightarrow [0, \infty]$ be σ-additive on $\mathcal{E}$. Then $\mu^* : \mathscr{P}(\Omega) \longrightarrow [0, \infty]$ defined by*

$$\mu^*(A) := \inf \Big\{ \textstyle\sum_{n \in \mathbb{N}} \mu(E_n) : (E_n)_n \subset \mathcal{E},\, A \subset \bigcup_{n \in \mathbb{N}} E_n \Big\} \qquad (A \in \mathscr{P}(\Omega))$$

is an outer measure. Moreover, $\sigma(\mathcal{E}) \subset \mathscr{M}(\mu^*)$ and $\mu^*|_{\mathcal{E}} = \mu$.

One may summarise these results in the following way: if a set function on a semi-ring $\mathcal{E}$ is σ-additive on $\mathcal{E}$ then it has a extension to a measure on $\sigma(\mathcal{E})$. If in addition Ω is σ-finite with respect to $\mathcal{E}$, then this extension is unique.

Sometimes, for instance in the construction of infinite products, it is convenient to work with the following criterion.

Lemma C.6. [Billingsley (1979), Thm. 10.2] *Let $\mathcal{E}$ be an algebra on a set Ω, and let $\mu : \mathcal{E} \longrightarrow [0, \infty)$ be a finitely additive set function with $\mu(\Omega) < \infty$. Then μ is σ-additive on $\mathcal{E}$ if and only if for each decreasing sequence $(A_n)_n \subset \mathcal{E}$, $A_n \searrow \emptyset$, one has $\mu(A_n) \to 0$.*

C.4 Measurable Functions and Mappings

Let (Ω, Σ) and (Ω', Σ') be measurable spaces. A mapping $\varphi : \Omega \longrightarrow \Omega'$ is called **measurable** if

$$[\varphi \in A] \in \Sigma \qquad (A \in \Sigma').$$

(It suffices to check this condition for each A from a generator of Σ'.) We denote by

$$\mathfrak{M}(\Omega;\Omega') = \mathfrak{M}(\Omega,\Sigma;\Omega,\Sigma')$$

the set of all measurable mappings between Ω and Ω'. For the special case $\Omega' = [0,\infty]$ we write

$$\mathfrak{M}_+(\Omega) := \bigl\{f : \Omega \longrightarrow [0,\infty] \, : \, f \text{ is measurable}\bigr\}.$$

Example: For $A \in \Sigma$ its characteristic function $\mathbf{1}_A$ is measurable, since one has $[\mathbf{1}_A \in B] = \emptyset, A, A^c, \Omega$, depending on whether or not 0 respectively 1 is contained in B.

Example: If Ω, Ω' are topological spaces and $\varphi : \Omega \longrightarrow \Omega'$ is continuous, then it is $\mathfrak{B}(\Omega) - \mathfrak{B}(\Omega')$ measurable.

Lemma C.7. [Lang (1993), p.117] *Let Ω' be a metric space and $\Sigma = \mathfrak{B}(\Omega')$ its Borel algebra. If $\varphi_n : \Omega \longrightarrow \Omega'$ is measurable for each $n \in \mathbb{N}$ and $\varphi_n \to \varphi$ pointwise, then φ is measurable as well.*

The following lemma summarises the basic properties of positive measurable functions.

Lemma C.8. [Billingsley (1979), Section 13] *Let (Ω, Σ, μ) be a measure space. Then the following assertions hold.*

a) *If $f, g \in \mathfrak{M}_+(\Omega), \alpha \geq 0$, then $fg, f+g, \alpha f \in \mathfrak{M}_+(\Omega)$.*
b) *If $f, g \in \mathfrak{M}(\Omega;\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then $fg, \alpha f + \beta g \in \mathfrak{M}(\Omega;\mathbb{R})$.*
c) *$f, g : \Omega \longrightarrow [-\infty,\infty]$ are measurable then $-f, \min\{f,g\}, \max\{f,g\}$ are measurable.*
d) *If $f_n : \Omega \longrightarrow [-\infty,\infty]$ is measurable for each $n \in \mathbb{N}$ then $\sup_n f_n, \inf_n f_n$ are measurable.*

A **simple function** on a measure space (Ω, Σ, μ) is a linear combination of characteristic functions of measurable sets. Positive measurable functions can be approximated by simple functions:

Lemma C.9. [Billingsley (1979), Thm. 13.5] *Let $f : \Omega \longrightarrow [0,\infty]$ be measurable. Then there exists a sequence of simple functions $(f_n)_n$ such that*

$$0 \leq f_n \leq f_{n+1} \nearrow f \qquad (\textit{pointwise as } n \to \infty).$$

If f is bounded, the convergence is uniform.

C.5 The Integral of Positive Measurable Functions

Given a measure space (Ω, Σ, μ) and a positive simple function

$$f = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{A_j}$$

on Ω, one defines its **integral** by

$$\int_\Omega f \,\mathrm{d}\mu := \sum_{j=1}^{n} \alpha_j \mu(A_j).$$

By using common refinements one can show that this definition is independent of the actual representation of f as a linear combination of characteristic functions. For a general $f \in \mathfrak{M}_+(\Omega)$ one defines

$$\int_\Omega f \,\mathrm{d}\mu := \lim_n \int_\Omega f_n \,\mathrm{d}\mu$$

where $(f_n)_n$ is an arbitrary sequence of simple functions with $0 \le f_n \nearrow f$ pointwise. (This is the way of [Bauer (1990), Chapter 11] and [Rana (2002), Section 5.2]; [Billingsley (1979), Section 15] takes a similar, but slightly different route.)

Theorem C.10. *The integral for positive measurable functions has the following properties.*

a) **(Action on Characteristic Functions)** *$(A \in \Sigma)$*

$$\int_\Omega \mathbf{1}_A \,\mathrm{d}\mu = \mu(A).$$

b) **(Additivity and homogeneity)** *$(f, g \in \mathfrak{M}_+(\Omega), \alpha \ge 0)$*

$$\int_\Omega (f + \alpha g) \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu + \alpha \int_\Omega g \,\mathrm{d}\mu.$$

c) **(Monotonicity)** *$(f, g \in \mathfrak{M}_+(\Omega))$*

$$f \le g \quad \Rightarrow \quad \int_\Omega f \,\mathrm{d}\mu \le \int_\Omega g \,\mathrm{d}\mu.$$

d) **(Beppo Levi, Monotone Convergence Theorem)** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathfrak{M}_+(\Omega)$ such that $0 \le f_1 \le f_2 \le \dots$ and $f_n \to f$ pointwise, then*

$$\int_\Omega f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu.$$

e) **(Fatou's lemma)** *Let $(f_n)_{n\in\mathbb{N}} \subset \mathfrak{M}_+(\Omega)$. Then*

$$\int_{\Omega} \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int_{\Omega} f_n \, d\mu.$$

Let $1 \le p \le \infty$. Then its **dual exponent** is the unique number $q = p' \in [1,\infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Theorem C.11 (Hölder's Inequality). *Let (Ω, Σ, μ) be a measure space, let $f, g \in \mathfrak{M}_+(\Omega)$, and let $1 < p < \infty$ with dual exponent q. Then $fg, f^p, g^q \in \mathfrak{M}_+(\Omega)$ as well and*

$$\int_{\Omega} fg \, d\mu \le \left(\int_{\Omega} f^p \, d\mu\right)^{1/p} \left(\int_{\Omega} g^q \, d\mu\right)^{1/q}.$$

See [Haase (2007)] for a nice proof.

C.6 Push-forward Measures and Measures with Density

If (Ω, Σ, μ) is a measure space, (Ω', Σ') is a measurable space and $\varphi : \Omega \longrightarrow \Omega'$ is measurable, then a measure is defined on Σ' by

$$[\varphi_* \mu](B) := \mu[\varphi \in B] \qquad (B \in \Sigma).$$

The measure $\varphi_*\mu$ is called the **image** of μ under φ, or the **push-forward** of μ along φ. If μ is finite or a probability measure, so is $\varphi_*\mu$. If $f \in \mathfrak{M}_+(\Omega')$ then

$$\int_{\Omega'} f \, d(\varphi_*\mu) = \int_{\Omega} (f \circ \varphi) \, d\mu.$$

Let (Ω, Σ, μ) be a measure space and $f \in \mathfrak{M}_+(\Omega)$. Then by

$$(f\mu)(A) := \int_A f \, d\mu := \int_{\Omega} \mathbf{1}_A f \, d\mu \qquad (A \in \Sigma)$$

a new measure $f\mu$ on Σ is defined. We call f the **density function** of $f\mu$. One has

$$\int_{\Omega} g \, d(f\mu) = \int_{\Omega} gf \, d\mu.$$

for all $g \in \mathfrak{M}_+(\Omega')$. [Billingsley (1979), Thm. 16.10 and 16.12].

Let μ, ν be two measures on Σ. We say that ν is **absolutely continuous** with respect to μ, written $\nu \ll \mu$, if $A \in \Sigma$, $\mu(A) = 0$ implies $\nu(A) = 0$. Clearly, if $\nu = f\mu$ with a density f, then ν is absolutely continuous with respect to μ. The converse is true under σ-finiteness conditions.

Theorem C.12 (Radon–Nikodym I). *Let (Ω, Σ, μ) be a σ-finite measure space, and let ν be a σ-finite measure on Σ, absolutely continuous with respect to μ. Then there is $f \in \mathfrak{M}_+(\Omega)$ such that $\nu = f\mu$.*

In [Billingsley (1979), Thm. 32.2] and [Bauer (1990), Satz 17.10] the proof is based on the so-called "Hahn decomposition" of signed measures; the Hilbert space approach of von Neumann is reproduced in [Rudin (1987), 6.10].

C.7 Product Spaces

If (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces, then on the product space $\Omega_1 \times \Omega_2$ we define the **product σ-algebra**

$$\Sigma_1 \otimes \Sigma_2 := \sigma\big\{A \times B : A \in \Sigma_1, B \in \Sigma_2\big\}.$$

If $\mathcal{E}_j$ is a generator of Σ_j with $\Omega_j \in \mathcal{E}_j$ for $j = 1, 2$, then

$$\mathcal{E}_1 \times \mathcal{E}_2 := \big\{A \times B : A \in \mathcal{E}_1, B \in \mathcal{E}_2\big\}$$

is a generator of $\Sigma_1 \otimes \Sigma_2$. As a consequence we obtain:

Lemma C.13. *Let Ω_1, Ω_2 be second countable topological (e.g., separable metric) spaces. Then*

$$\mathfrak{B}(\Omega_1 \otimes \Omega_2) = \mathfrak{B}(\Omega_1) \otimes \mathfrak{B}(\Omega_2).$$

If (Ω, Σ) is another measurable space, then a mapping $f = (f_1, f_2) : \Omega \longrightarrow \Omega_1 \times \Omega_2$ is measurable if and only if the projections $f_1 = \pi_1 \circ f, f_2 = \pi_2 \circ f$ are both measurable.

If $f : (\Omega_1 \times \Omega_2, \Sigma_1 \otimes \Sigma_2) \longrightarrow (\Omega', \Sigma')$ is measurable, then $f(x, \cdot) : \Omega_2 \longrightarrow \Omega'$ is measurable, for every $x \in \Omega_1$, see [Billingsley (1979), Theorem 18.1].

Theorem C.14 (Tonelli). [Billingsley (1979), Theorem 18.3] *Let $(\Omega_j, \Sigma_j, \mu_j)$, $j = 1, 2$, be σ-finite measure spaces and $f \in \mathfrak{M}_+(\Omega_1 \times \Omega_2)$. Then the functions*

$$\begin{aligned} F_1 :& \Omega_1 \longrightarrow [0, \infty], \quad x \longmapsto \int_{\Omega_2} f(x, \cdot)\, \mathrm{d}\mu_2 \\ F_2 :& \Omega_2 \longrightarrow [0, \infty], \quad y \longmapsto \int_{\Omega_1} f(\cdot, y)\, \mathrm{d}\mu_1 \end{aligned}$$

are measurable and there is a unique measure $\mu_1 \otimes \mu_2$ such that

$$\int_{\Omega_1} F_1\, \mathrm{d}\mu_1 = \int_{\Omega_1 \times \Omega_2} f\, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_2} F_2\, \mathrm{d}\mu_2.$$

The measure $\mu_1 \otimes \mu_2$ is called the **product measure** of μ_1, μ_2. Note that for the particular case $F = f_1 \otimes f_2$, with

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1) \cdot f_2(x_2) \qquad (f_j \in \mathfrak{M}_+(\Omega_j), x_j \in \Omega_j \quad (j = 1, 2)),$$

we obtain

$$\int_{\Omega_1\times\Omega_2}(f_1\otimes f_2)\,\mathrm{d}(\mu_1\otimes\mu_2)=\left(\int_{\Omega_1}f_1\,\mathrm{d}\mu_1\right)\left(\int_{\Omega_2}f_2\,\mathrm{d}\mu_2\right).$$

Infinite Products and Ionescu Tulcea's Theorem

For a measurable space (Ω,Σ) we denote by $\mathrm{M}^+(\Omega,\Sigma)$ the set of all positive and by $\mathrm{M}^1(\Omega,\Sigma)$ the set of all probability measures on (Ω,Σ). There is a natural σ-algebra $\tilde{\Sigma}$ on $\mathrm{M}^+(\Omega,\Sigma)$, the smallest such that each mapping

$$\mathrm{M}^+(\Omega,\Sigma)\longrightarrow[0,\infty],\quad \nu\longmapsto\nu(A)\qquad(A\in\Sigma)$$

is measurable.

Let (Ω_j,Σ_j), $j=1,2$ be measurable spaces. A **measure kernel** from Ω_1 to Ω_2 is a measurable mapping $\mu:\Omega_2\longrightarrow\mathrm{M}^+(\Omega_1,\Sigma_1)$. Such a kernel μ can also be interpreted as a mapping of two variables

$$\mu:\Omega_2\times\Sigma\longrightarrow[0,\infty],$$

and we shall do so when it seems convenient. If $\mu(y,\cdot)\in\mathrm{M}^1(\Omega_1,\Sigma_1)$ for each $y\in\Omega_2$ then μ is called a **probability kernel**.

Let (Ω,Σ) be another measurable space, and let $\mu:\Omega_2\longrightarrow\mathrm{M}^+(\Omega_1,\Sigma_1)$ be a kernel. Then there is an induced operator

$$T_\mu:\mathfrak{M}_+(\Omega\times\Omega_1)\longrightarrow\mathfrak{M}_+(\Omega\times\Omega_2)$$
$$(T_\mu f)(x,x_2):=\int_{\Omega_1}f(x,x_1)\,\mu(x_2,\mathrm{d}x_1)\quad(x\in\Omega,x_2\in\Omega_2).$$

The operator T_μ is additive and positively homogeneous, and if $f_n\nearrow f$ pointwise on Ω_1 then $T_\mu f_n\nearrow T_\mu f$ pointwise on Ω_2. Moreover,

$$T_\mu(f\otimes g)=(f\otimes\mathbf{1})\cdot T_\mu(\mathbf{1}\otimes g)\qquad(f\in\mathfrak{M}_+(\Omega),g\in\mathfrak{M}_+(\Omega_2)).$$

Conversely, each operator $T:\mathfrak{M}_+(\Omega\times\Omega_1)\longrightarrow\mathfrak{M}_+(\Omega\times\Omega_2)$ with these properties is of the form T_μ, for some kernel μ.

If $\mu:\Omega_2\longrightarrow\mathrm{M}^+(\Omega_1)$ and $\nu:\Omega_3\longrightarrow\mathrm{M}^+(\Omega_2)$ are kernels, then $T_\nu\circ T_\mu=T_\eta$ for

$$\eta(x_3,A):=\int_{\Omega_2}\mu(x_2,A)\nu(x_3,\mathrm{d}x_2)\qquad(x_3\in\Omega_3,A\in\Sigma_1).$$

Kernels can be used to construct measures on infinite products. Let (Ω_n,Σ_n), $n\in\mathbb{N}$, be measurable spaces, and let $\Omega:=\prod_{n\in\mathbb{N}}\Omega_n$ be the Cartesian product, with the projections $\pi_n:\Omega\longrightarrow\Omega_n$. The natural σ-algebra on Ω is

$$\bigotimes_n\Sigma_n:=\sigma\big\{\pi_n^{-1}(A_n):n\in\mathbb{N},A_n\in\Sigma_n\big\}.$$

A generating algebra is

$$\mathscr{A} := \Big\{ A_n \times \prod_{k>n} \Omega_k \,:\, n \in \mathbb{N}, A_n \in \Sigma_1 \otimes \ldots \otimes \Sigma_n \Big\},$$

the algebra of **cylinder sets**.

Theorem C.15 (Ionescu Tulcea). [Ethier and Kurtz (1986), p.504] *Let* (Ω_n, Σ_n)*,* $n \in \mathbb{N}$*, be measurable spaces, let*

$$\mu_n : \Omega_1 \times \cdots \times \Omega_{n-1} \longrightarrow \mathrm{M}^1(\Omega_n) \qquad (n \in \mathbb{N}, n \geq 2)$$

be probability kernels, and let μ_1 *be a probability measure on* Ω_1*. Let*

$$X^{(n)} := \Omega_1 \times \cdots \times \Omega_n \qquad \textit{with} \qquad \Sigma^{(n)} = \Sigma_1 \otimes \ldots \otimes \Sigma_n.$$

Let, for $n \geq 1$*,* $T_n : \mathfrak{M}_+(X^{(n)}, \Sigma^{(n)}) \longrightarrow \mathfrak{M}_+(X^{(n-1)}, \Sigma^{(n-1)})$ *be given by*

$$(T_n f)(x^{(n-1)}) = \int_{\Omega_n} f(x^{(n-1)}, x_n)\, \mu_n(x^{(n-1)}, \mathrm{d}x_n) \qquad (x^{(n-1)} \in X^{(n-1)}).$$

Then there is a unique probability measure ν *on* $X^{(\infty)} := \prod_{n\in\mathbb{N}} \Omega_n$ *such that*

$$\int_{X^{(\infty)}} f(x_1, \ldots, x_n)\, \mathrm{d}\nu(x_1, \ldots) = T_1 T_2 \ldots T_n f \qquad (f \in \mathfrak{M}_+(\Omega_1 \times \cdots \times \Omega_n))$$

for every $n \in \mathbb{N}$*.*

An important special case of the Ionescu Tulcea theorem is the construction of the infinite product measure. Here one has a probability measure ν_n on (Ω_n, Σ_n), for each $n \in \mathbb{N}$. If one applies the Ionescu Tulcea theorem with $\mu_n \equiv \nu_n$, then the ν of the theorem satisfies

$$(\pi_1, \ldots, \pi_n)_* \nu = \nu_1 \otimes \ldots \otimes \nu_n \qquad (n \in \mathbb{N}).$$

We write $\nu := \bigotimes_n \nu_n$ and call it the **product** of the ν_n. For products of uncountably many probability spaces see [Hewitt and Stromberg (1969), Chapter 22].

C.8 Null Sets

Let (Ω, Σ, μ) be a measure space. A set $A \subset \Omega$ is called a **null set** or **negligible** if there is a set $N \in \Sigma$ such that $A \subset N$ and $\mu(N) = 0$. (In general a null set need not be measurable). Null sets have the following properties:

a) If A is a null set and $B \subset A$ then B is also a null set.
b) If A_n is a null set ($n \in \mathbb{N}$), then $\bigcup_n A_n$ is a null set.

Lemma C.16. [Billingsley (1979), Theorem 15.2] *Let* (Ω, Σ, μ) *be a measure space and let* $f : \Omega \longrightarrow [-\infty, \infty]$ *be measurable. Then the following assertions hold.*

a) $\int_\Omega |f|\,\mathrm{d}\mu = 0$ *if and only if the set* $[\,f \neq 0\,] = [\,|f| > 0\,]$ *is a null set.*

b) *If* $\int_\Omega |f|\,\mathrm{d}\mu < \infty$*, then the set* $[\,|f| = \infty\,]$ *is a null set.*

One says that two functions f,g are equal μ**-almost everywhere** (abbreviated by "$f = g$ a.e." or "$f \sim_\mu g$") if the set $[\,f \neq g\,]$ is a null set. More generally, let P be a property of points of Ω. Then P is said to hold **almost everywhere** or for μ**-almost all** $x \in \Omega$ if the set

$$\{x \in \Omega \,:\, P \text{ does not hold for } x\}$$

is a μ-null set. If μ is understood, we leave out the reference to it.

For each set Ω', the relation $\sim_\mu$ ("is equal μ-almost everywhere to") is an equivalence relation on the space of mappings from Ω to Ω'. For such a mapping f we sometimes denote by $[f]$ its equivalence class, in situations when notational clarity is needed. If μ is understood, we write simply $\sim$ instead of $\sim_\mu$. By choosing $\Omega = \{0,1\}$ an equivalence relation on Σ is induced via

$$A \sim B \quad \overset{\text{Def.}}{\iff} \quad \mathbf{1}_A = \mathbf{1}_B \quad \mu\text{-a.e.} \quad \iff \quad \mu(A \triangle B) = 0.$$

The space of equivalence classes $\Sigma/\sim$ is called the **measure algebra**. For a set $A \in \Sigma$ we sometimes write $[\,A\,]$ for its equivalence class with respect to $\sim$, but usually we omit the brackets and simply write A again. Clearly, if $f = g$ μ-a.e. then $[\,f \in B\,] \sim [\,g \in B\,]$ for every $B \subset \Omega'$. The usual set-theoretic operations can be induced on the elements of $\Sigma/\sim$ by setting

$$[A] \cap [B] := [A \cap B], \quad \bigcup\nolimits_n [A_n] := \big[\bigcup\nolimits_n A_n\big] \;\ldots.$$

Also, one defines

$$\mu[A] := \mu(A) = \int_\Omega \mathbf{1}_A \,\mathrm{d}\mu \qquad (A \in \Sigma)$$

and writes $\emptyset := [\emptyset]$ again. Hence on the measure algebra, $\mu(A) = 0$ if and only if $A = \emptyset$.

C.9 Convergence in Measure

Let (Ω, Σ, μ) be a σ-finite measure space and (X,d) a complete metric space, with its Borel σ-algebra. Let

$$\mathfrak{M}_s(\Omega,\Sigma;X) := \{f \in \mathfrak{M}(\Omega;X) \,:\, f(\Omega) \text{ is separable}\}.$$

Note that $\mathfrak{M}_s(\Omega;X) = \mathfrak{M}(\Omega;X)$ if X is separable. Choose a complete metric d on X such that d induces the topology and such that d is uniformly bounded. (For example, if d is *any* complete metric inducing the topology, one can replace d by $d/(d+1)$ to obtain an equivalent metric which is also bounded.) Using Lemma C.13 one sees that the mapping

$$d(f,g) : \Omega \times \Omega \longrightarrow [0,\infty), \quad (x,y) \longmapsto d(f(x),g(y))$$

is product measurable. For a fixed $A \in \Sigma$ with $\mu(A) < \infty$ we define a semi-metric on $\mathfrak{M}_s(\Omega;X)$ by

$$d_A(f,g) := \int_A d(f,g)\,\mathrm{d}\mu \qquad (f,g \in \mathfrak{M}_s(\Omega;X)).$$

Clearly $d_A(f,g) = 0$ if and only if $f = g$ almost everywhere on A. One has $f_n \to f$ with respect to d_A if and only if

$$\mu\big([d(f_n,f) > \varepsilon] \cap A\big) \to 0 \quad \text{for each } \varepsilon > 0.$$

Convergence in d_Ω is called **convergence globally in measure**.

Let (Ω,Σ,μ) be a σ-finite measure space, and choose $\Omega_n \in \Sigma$ of finite measure and such that $\Omega = \bigcup_n \Omega_n$. Let

$$D(f,g) := \textstyle\sum_{n=1}^{\infty} 2^{-n} d_{\Omega_n}(f,g) \qquad (f,g \in \mathfrak{M}_s(\Omega;X)).$$

Then D is a semi-metric on $\mathfrak{M}_s(\Omega;X)$. The convergence with respect to D is called **convergence (locally) in measure**. Note that $D = d_\Omega$ if μ is finite.

Theorem C.17. *Let (Ω,Σ,μ) be a σ-finite measure space and X a completely metrizable space.*

a) *The semi-metric D on $\mathfrak{M}_s(\Omega;X)$ is complete.*
b) *$D(f,g) = 0$ if and only if $f = g$ μ-almost everywhere.*
c) *$f_n \to f$ in measure if and only if every subsequence of $(f_n)_n$ has a subsequence which converges to f pointwise almost everywhere.*
d) *$D(f_n,f) \to 0$ if and only if $d_A(f_n,f) \to 0$ for all $A \in \Sigma$, $\mu(A) < \infty$.*

Note that c) shows that a choice of an equivalent (complete bounded) metric on E leads to an equivalent semi-metric on $\mathfrak{M}_s(\Omega;E)$. We do not know of a good reference for Theorem C.17. In [Bauer (1990), Chap. 20] one finds all decisive details, although formulated for the case $E = \mathbb{R}$. The case of a probability space is treated in [Kallenberg (2002), Lemmas 4.2 and 4.6].

Theorem C.18 (Egoroff). [Rana (2002), 8.2.4] *Let (Ω,Σ,μ) be a finite measure space and X a complete metric space. Let $(f_n)_n \subset \mathfrak{M}(\Omega;X)$ and $f : \Omega \longrightarrow X$. Then $f_n \to f$ pointwise almost everywhere if and only if for each $\varepsilon > 0$ there is $A \in \Sigma$ with $\mu(A^{\mathrm{c}}) < \varepsilon$ and $f_n \to f$ uniformly on A.*

We denote by

$$\mathrm{L}^0(\Omega;X) := \mathrm{L}^0(\Omega,\Sigma,\mu;X) := \mathfrak{M}_s(\Omega;X)/\!\sim$$

the space of equivalence classes of measurable, separably-valued mappings modulo equality almost everywhere. By a) and b) of the theorem above, D induces a complete metric on $\mathrm{L}^0(\Omega;X)$.

By restricting to characteristic functions, i.e., to the case $X = \{0,1\}$, this induces a (complete!) metric on the measure algebra $\Sigma/\!\sim$. If $\mu(\Omega) = 1$, this metric is given by

$$d([A],[B]) = d_\Omega(\mathbf{1}_A, \mathbf{1}_B) = \mu(A \triangle B) \qquad (A, B \in \Sigma).$$

C.10 The Lebesgue-Bochner Spaces

Let (Ω, Σ, μ) be a σ-finite measure space and X be a Banach space with norm $\|\cdot\|_X$. Then $\mathrm{L}^0(\Omega;X)$ is an F-space, i.e., a topological vector space, completely metrisable by a translation invariant metric. A function $f : \Omega \longrightarrow X$ is called a **step function** if it is of the form

$$f = \sum_{j=1}^{n} \mathbf{1}_{A_j} \otimes x_j = \sum_{j=1}^{n} \mathbf{1}_{A_j}(\cdot) x_j$$

for some finitely many $x_j \in X$, $A_j \in \Sigma$, $\mu(A_j) < \infty$ $(j = 1, \dots, n)$. We denote by

$$\mathrm{St}(\Omega;X) := \mathrm{lin}\{\mathbf{1}_A \otimes x : x \in X,\ A \in \Sigma,\ \mu(A) < \infty\}$$

the space of all X-valued step functions. An X-valued function is called **μ-measurable** if there is a sequence of step functions converging to f pointwise μ-almost everywhere.

Lemma C.19. [Lang (1993), pp. 124 and 142] *Let (Ω, Σ, μ) be a σ-finite measure space, let X be a Banach space, and let $f : \Omega \longrightarrow X$ be a mapping. Then $[f] \in \mathrm{L}^0(\Omega;X)$ if and only if f is μ-measurable, if and only if there is a sequence $(f_n)_n \subset \mathrm{St}(\Omega;X)$ of step functions such that $f_n \to f$ a.e. and $\|f_n(\cdot)\|_X \le 2\,\|f(\cdot)\|_X$ a.e., for all $n \in \mathbb{N}$.*

A consequence of this lemma together with Theorem C.17 is that $\mathrm{St}(\Omega;X)$ is dense in the complete metric space $\mathrm{L}^0(\Omega;X)$.

For $f \in \mathrm{L}^0(\Omega;X)$ we define

$$\|f\|_\infty := \inf\{t > 0 : \mu[\,\|f(\cdot)\|_X > t\,] = 0\}$$

and we set

$$\mathrm{L}^\infty(\Omega;X) := \mathrm{L}^\infty(\Omega,\Sigma,\mu;X) := \{f \in \mathrm{L}^0(\Omega;X) : \|f\|_\infty < \infty\}.$$

Then $\|\cdot\|_\infty$ defines a complete norm on $\mathrm{L}^\infty(\Omega;X)$. We simply write $\mathrm{L}^\infty(\Omega)$ when we deal with scalar-valued functions.

Let $1 \le p < \infty$. For $f \in \mathrm{L}^0(\Omega;X)$ we define

$$\|f\|_p := \left(\int_\Omega \|f(\cdot)\|_X^p \,\mathrm{d}\mu\right)^{\frac{1}{p}}$$

and $\mathrm{L}^{\mathrm{p}}(\Omega;X) := \mathrm{L}^{\mathrm{p}}(\Omega,\Sigma,\mu;X) := \{f \in \mathrm{L}^0(\Omega;X) : \|f\|_p < \infty\}$. We simply write $\mathrm{L}^{\mathrm{p}}(\Omega)$ when dealing with scalar-valued functions.

Theorem C.20. *Let (Ω,Σ,μ) be a σ-finite measure space, X a Banach space and $1 \le p < \infty$. Then the following assertions hold.*

a) *$\|\cdot\|_p$ is a complete norm on $\mathrm{L}^{\mathrm{p}}(\Omega;X)$.*

b) *The embedding $\mathrm{L}^{\mathrm{p}}(\Omega;X) \subset \mathrm{L}^0(\Omega;X)$ is continuous.*

c) *If $f_n \to f$ in $\mathrm{L}^{\mathrm{p}}(\Omega;X)$ then there is $g \in \mathrm{L}^{\mathrm{p}}(\Omega;\mathbb{R})$ and a subsequence $(f_{n_k})_k$ such that $\|f_{n_k}(\cdot)\|_X \le g$ a.e., for all $k \in \mathbb{N}$, and $f_{n_k} \to f$ pointwise a.e..*

d) *$\mathrm{St}(\Omega;X)$ is dense in $\mathrm{L}^{\mathrm{p}}(\Omega;X)$.*

e) **(LDC)** *If $(f_n)_n \subset \mathrm{L}^{\mathrm{p}}(\Omega;X)$ $f_n \to f$ in measure and there is $g \in \mathrm{L}^{\mathrm{p}}(\Omega;\mathbb{R})$ such that $\|f_n(\cdot)\|_X \le g$ a.e., for all $n \in \mathbb{N}$, then $f \in \mathrm{L}^{\mathrm{p}}(\Omega;X)$, and $\|f_n - f\|_p \to 0$.*

The abbreviation "LDC" stands for *L*ebesgue's *D*ominated *C*onvergence theorem.

The (Bochner-)Integral

We want to integrate functions from $\mathrm{L}^1(\Omega,\Sigma,\mu;X)$. In the case $X = \mathbb{C}$ one can use the already defined integral for positive measurable functions, and this is how it is done in most of the textbooks. However, this does not work for Banach space-valued functions. Therefore we take a different route and shall see eventually that in the case $X = \mathbb{C}$ we recover the common definition.

For a step function $f = \sum_{j=1}^n \mathbf{1}_{A_j} \otimes x_j$ we define its **integral** by

$$\int_\Omega f \,\mathrm{d}\mu := \sum_{j=1}^n \mu(A_j)x_j.$$

This is independent of the representation of f and hence defines a linear mapping

$$\left[f \longmapsto \int_\Omega f \,\mathrm{d}\mu\right] : \mathrm{St}(\Omega;X) \longrightarrow X.$$

Since obviously

$$\left\| \int_\Omega f \mathrm{d}\mu \right\|_X \le \int_\Omega \|f(\cdot)\|_X \,\mathrm{d}\mu = \|f\|_1 \qquad (f \in \mathrm{St}(\Omega;X)),$$

this mapping can be extended by continuity to all of $\mathrm{L}^1(\Omega;X)$ to a linear contraction

$$\left[f \longmapsto \int_\Omega f \,\mathrm{d}\mu\right] : \mathrm{L}^1(\Omega;X) \longrightarrow X.$$

It is easy to see that for $f : \Omega \longrightarrow [0,\infty)$ this definition of the integral and the one for positive measurable functions coincide. This shows that for complex-valued functions our definition of the integral leads to the same as the one usually given in more elementary treatments.

If Y is another Banach space and $T : X \longrightarrow Y$ is a bounded linear mapping, then

$$\int_\Omega (T \circ f)\,\mathrm{d}\mu = T \int_\Omega f\,\mathrm{d}\mu \qquad (f \in \mathrm{L}^1(\Omega;X)).$$

Applying linear functionals yields

$$\Big\| \int_\Omega f\,\mathrm{d}\mu \Big\|_X \le \int_\Omega \|f(\cdot)\|_X\,\mathrm{d}\mu \qquad (f \in \mathrm{L}^1(\Omega;X)).$$

Theorem C.21 (Averaging Theorem). [Lang (1993), Thm. 5.15] *Let $S \subset X$ be a closed subset, and let $f \in \mathrm{L}^1(\Omega;X)$. If*

$$\frac{1}{\mu(A)} \int_A f\,\mathrm{d}\mu \in S$$

for all $A \in \Sigma$ such that $0 < \mu(A) < \infty$, then $f \in S$ almost everywhere.

As a corollary one obtains that if $\int_A f = 0$ for all A with finite measure, then $f = 0$ almost everywhere.

C.11 Approximations

Let (Ω, Σ, μ) be a measure space. Directly from Lemma C.9 we see that the set of simple functions is dense in $\mathrm{L}^\infty(\Omega, \Sigma, \mu; \mathbb{R})$, and we know already that $\mathrm{St}(\Omega, \Sigma; X)$ is dense in $\mathrm{L}^p(\Omega;X)$ if X is a Banach space and $p < \infty$. Here we are interested in more refined statements, involving step functions

$$\mathrm{St}(\Omega, \mathcal{E}; X) := \mathrm{lin}\big\{ \mathbf{1}_B \otimes x : B \in \mathcal{E}, x \in X \big\}$$

with respect to a generator $\mathcal{E}$ of Σ.

Lemma C.22. [Billingsley (1979), Thm. 11.4] *Let $\mathcal{E} \subset \Sigma$ be a ring with $\sigma(\mathcal{E}) = \Sigma$. Fix $C \in \mathcal{E}$ with $\mu(C) < \infty$ and define*

$$\mathcal{E}_C := \big\{ B \in \mathcal{E} : B \subset C \big\} = \big\{ B \cap C : B \in \mathcal{E} \big\}.$$

Then for each $A \in \Sigma$ and each $\varepsilon > 0$ there is $B \in \mathcal{E}_C$ such that $\mu((A \cap C) \triangle B) < \varepsilon$.

Based on the lemma, one can prove the following.

Theorem C.23. *Let (Ω, Σ, μ) be a measure space and let $\mathcal{E} \subset \Sigma$ be a ring that generates Σ and consists exclusively of sets of finite measure. Furthermore, suppose that Ω is σ-finite with respect to $\mathcal{E}$. Then the following assertions hold.*

a) $\{[B] : B \in \mathcal{E}\}$ *is dense in* $\Sigma/\!\sim$.

b) *If X is a Banach space then* $\mathrm{St}(\Omega,\mathcal{E};X)$ *is dense in* $\mathrm{L}^0(\Omega;X)$.

c) *If X is a Banach space and* $1 \le p < \infty$ *then* $\mathrm{St}(\Omega,\mathcal{E};X)$ *is dense in* $\mathrm{L}^{\mathrm{p}}(\Omega;X)$.

Fubini's Theorem

As an application we consider two σ-finite measure spaces $(\Omega_j,\Sigma_j,\mu_j)$, $j=1,2$, and their product

$$(\Omega,\Sigma,\mu)=(\Omega_1\times\Omega_2,\Sigma_1\otimes\Sigma_2,\mu_1\otimes\mu_2).$$

Let $\mathcal{R}:=\{A_1\times A_2 \,:\, A_j\in\Sigma_j,\mu(A_j)<\infty\ (j=1,2)\}$ be the set of **measurable rectangles**. Then $\mathcal{R}$ is a semi-ring, and its generated ring $\mathcal{E}$ satisfies the conditions of Theorem C.23. Since $\mathcal{E}$ consists of disjoint unions of members of $\mathcal{R}$, we obtain:

Corollary C.24. *Let X be a Banach space and* $1\le p<\infty$. *The space*

$$\operatorname{lin}\bigl\{\mathbf{1}_{A_1}\otimes\mathbf{1}_{A_2}\otimes x \;:\; x\in X, A_j\in\Sigma_j,\mu(A_j)<\infty\ (j=1,2)\bigr\}$$

is dense in $\mathrm{L}^{\mathrm{p}}(\Omega;X)$.

Using this and Tonelli's theorem, one proves Fubini's theorem.

Theorem C.25 (Fubini). [Lang (1993), Thm. 8.4] *Let X be a Banach space and* $f\in\mathrm{L}^1(\Omega_1\times\Omega_2;X)$. *Then for μ_1-almost every $x\in\Omega_1$,* $f(x,\cdot)\in\mathrm{L}^1(\Omega_2;X)$ *and with*

$$F:=\left(x\longmapsto\int_{\Omega_2} f(x,\cdot)\,\mathrm{d}\mu_2\right)$$

(defined almost everywhere on Ω_1) one has $F\in\mathrm{L}^1(\Omega_1;X)$*; moreover,*

$$\int_{\Omega_1} F\,\mathrm{d}\mu_1=\int_{\Omega_1}\int_{\Omega_2} f(x,y)\,\mathrm{d}\mu_2(y)\,\mathrm{d}\mu_1(x)=\int_{\Omega_1\otimes\Omega_2} f\,\mathrm{d}(\mu_1\otimes\mu_2).$$

C.12 Complex Measures

A **complex measure** on a measurable space (Ω,Σ) is a mapping $\mu:\Sigma\longrightarrow\mathbb{C}$ which is σ-additive and satisfies $\mu(\emptyset)=0$. If the range of μ is contained in $\mathbb{R}$, μ is called a **signed measure**. For a complex measure μ one defines its **total variation** $|\mu|$ by

$$|\mu|(A):=\inf\left\{\sum_{n=1}^{\infty}|\mu(A_n)| \,:\, (A_n)_n\subset\Sigma \text{ pairwise disjoint}, A=\bigcup_n A_n\right\}$$

for $A\in\Sigma$. Then $|\mu|$ is a positive finite measure, see [Rudin (1987), Thm. 6.2]. With respect to the norm $\|\mu\|_1:=|\mu|(\Omega)$, the space of complex measures on (Ω,Σ) is a Banach space.

Let μ be a complex measure on a measurable space (Ω,Σ), and let X be a Banach space. For a step function $f=\sum_{j=1}^{n}\mathbf{1}_{A_j}\otimes x_j\in \mathrm{St}(\Omega,\Sigma,|\mu|;X)$ one defines

$$\int_\Omega f\,\mathrm{d}\mu = \sum_{j=1}^{n}\mu(A_j)x_j$$

as usual, and shows (using finite additivity) that this does not depend on the representation of f. Moreover, one obtains

$$\Big\|\int_\Omega f\,\mathrm{d}\mu\Big\|_X \le \int_\Omega \|f(\cdot)\|_X\,\mathrm{d}|\mu| = \|f\|_{\mathrm{L}^1(|\mu|)}\,,$$

whence the integral has a continuous linear extension to all of $\mathrm{L}^1(\Omega,\Sigma,|\mu|;X)$.

Let (Ω,Σ,μ) be a measure space. Then for $f\in\mathrm{L}^1(\Omega;\mathbb{C})$ by

$$(f\mu)(A) := \int_\Omega \mathbf{1}_A f\,\mathrm{d}\mu \qquad (A\in\Sigma)$$

a complex measure is defined, with $|f\mu| = |f|\,\mu$.

Theorem C.26 (Radon–Nikodym II). [Rudin (1987), Thm. 6.10] *Let (Ω,Σ,μ) be a σ-finite measure space. The mapping $(f\longmapsto f\mu)$ is an isometric isomorphism between $\mathrm{L}^1(\Omega;\mathbb{C})$ and the space of complex measures ν on Σ with the property that $|\nu|$ is absolutely continuous with respect to μ.*

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