Appendix A
Topology

A.1 Metric spaces

A **metric space** is a pair (Ω, d) consisting of a non-empty set Ω and a function $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ which describes the distance between any two points of Ω, and for which we require the following properties:

(i) $d(x,y) \geq 0$, and $d(x,y) = 0$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$ **(triangle inequality)**.

The function d is called a **metric** on Ω. If instead of (i) we require only $d(x,x) = 0$ for all $x \in \Omega$, we obtain the notion of a **semi-metric**. For $A \subset \Omega$ and $x \in \Omega$ we define

$$d(x,A) := \inf\{d(x,y) \,:\, y \in A\}$$

called the **distance** of x from A. The triangle inequality implies that

$$d(x,A) - d(y,A) \leq d(x,y) \qquad (x, y \in \Omega).$$

By a **ball** with center x and radius $r > 0$ we mean either of the sets

$$\mathrm{B}(x,r) := \{y \in \Omega \,:\, d(x,y) < r\},$$
$$\overline{\mathrm{B}}(x,r) := \{y \in \Omega \,:\, d(x,y) \leq r\}.$$

A set $O \subseteq X$ is called **open** if for all $x \in O$ there is a ball $B \subseteq O$ with centre x and radius $r > 0$. A set $A \subseteq \Omega$ is called **closed** if $\Omega \setminus A$ is open. The ball $\mathrm{B}(x,r)$ is open, and the ball $\overline{\mathrm{B}}(x,r)$ is closed for any $x \in \Omega$ and $r > 0$. Two trivially open, and at the same time closed sets are the empty set $\emptyset$ and the set Ω itself.

A sequence (x_n) in Ω is **convergent to the limit** $x \in \Omega$ (we write: $x_n \to x$), if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $d(x_n, x) < \varepsilon$ for $n \geq n_0$. For a subset $A \subset \Omega$ the following assertions are equivalent:

(i) A is closed;

(ii) if $x \in \Omega$ and $d(x,A) = 0$ then $x \in A$;

(iii) if $(x_n)_n \subset A$ and $x_n \to x \in \Omega$, then $x \in A$.

A **Cauchy-sequence** (x_n) in Ω is a sequence with the property that for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $d(x_n, x_m) < \varepsilon$ for $n, m \geq n_0$. A convergent sequence is always a Cauchy-sequence. A metric space is called **complete** if the converse implication also holds.

A.2 Topological spaces

Starting with a non-empty set Ω we would like to define open sets. For this purpose observe that open sets in Euclidean spaces or as defined in A.1 satisfy the following:

a) $\emptyset$ and Ω are open.

b) If O_1 and O_2 are open, so is their intersection $O_1 \cap O_2$.

c) An arbitrary union of open sets is open.

We now take these three properties as the defining characteristics for the family of open sets in Ω. More precisely, assume that $\mathscr{O} \subseteq \mathfrak{P}(\Omega)$ satisfies

(i) $\emptyset, \Omega \in \mathscr{O}$;

(ii) If $O_1, \ldots, O_n \in \mathscr{O}$, then $O_1 \cap \cdots \cap O_n \in \mathscr{O}$; (i.e., $\mathscr{O}$ is $\cap$-stable,

(iii) If $O_\iota \in \mathscr{O}$, $\iota \in I$, then $\bigcup_{\iota \in I} O_\iota \in \mathscr{O}$.

We call $\mathscr{O}$ the **family of open sets** in Ω, and we say that $(\Omega, \mathscr{O})$ (or simply Ω) is a **topological space**. We also call $\mathscr{O}$ itself the topology on Ω. **Closed sets** are then the complements of open sets. Finite unions and arbitrary intersections of closed sets are closed.

A topological space is called **metrisable** if there exists a metric that induces the topology. Not every topological space is metrisable, cf. Section A.9 below.

If $\mathscr{O}, \mathscr{O}'$ are both topologies on Ω and $\mathscr{O}' \subseteq \mathscr{O}$, then we say that the topology $\mathscr{O}$ is **finer** than $\mathscr{O}'$ (or $\mathscr{O}'$ is **coarser** than $\mathscr{O}$).

Example A.1. a) Let Ω be non-empty, and $\mathscr{O} := \{\emptyset, \Omega\}$. Then $\mathscr{O}$ satisfies the properties (i)–(iii), thus $(\Omega, \mathscr{O})$ is a topological space, whose topology is called the **trivial topology**. Note that it is the coarsest among all topologies on Ω.

b) The other extreme case is when we set $\mathscr{O} := \mathfrak{P}(\Omega)$, the largest possible choice. The so defined topology is the **discrete topology**. The explanation for this terminology is that for all points in $x \in \Omega$ the singleton $\{x\}$ is open (and also closed). The discrete topology is the finest among all possible topologies on Ω. In this case all sets are closed and open at the same time.

c) If (Ω, d) is a metric space, the open sets from A.1 define a topology $\mathscr{O}_d$ on Ω, and we say that the metric induces the topology on Ω. In this case there are many different metrics that induce the same topology, and these metrics we call **equivalent metrics**.

d) For Ω is a non-empty set, the function $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ defined by $d(x,y) = 0$ for $x = y$, $d(x,y) = 1$ for $x \neq y$ is a metric and it induces the discrete topology.

A **neighbourhood** of a point $x \in \Omega$ is a set U such that there is an open set $O \subseteq \Omega$ with $x \in O \subseteq U$. An open set is a neighbourhood of all of its points. If A is a neighbourhood of x, then x is called an interior point of A. The set of all interior points of A is denoted by $\mathring{A}$ and is called the **interior** of A. The **closure** $\overline{A}$ of a subset $A \subseteq \Omega$ is

$$\bigcap_{\substack{A \subseteq F \subseteq \Omega \\ F \text{ closed}}} F,$$

which is obviously the smallest closed set that contains A. If (Ω, d) is a metric space and $A \subset \Omega$, then $\overline{A} = \{x : d(x,A) = 0\}$, and $x \in \overline{A}$ iff x is the limit of a sequence in A.

To define a topology it is not necessary to specify all the open sets. We may as well proceed similarly to the metric case, by replacing the family of open balls

$$\{\mathrm{B}(x,r) : x \in \Omega, \, r > 0\}$$

by a suitable system of neighbourhoods. A **base** $\mathscr{B} \subseteq \mathscr{O}$ for the topology $\mathscr{O}$ on Ω, is a system which has the property that all open sets can be written as the union of base-elements. For example, the family of open balls is a base for the topology induced by the metric on the metric space (Ω, d). A topological space is called **second countable** if it has a countable base for its topology.

A topological space Ω is called **Hausdorff** if any two points $x, y \in \Omega$ can be separated by disjoint open neighbourhoods, i.e., there are $U, V \in \mathscr{O}$ with $U \cap V = \emptyset$ and $x \in U$, $y \in V$. In a Hausdorff space a singleton $\{x\}$ $x \in \Omega$ is closed. Discrete spaces are Hausdorff, while trivial topological spaces are not unless Ω is a singleton. More generally metric spaces are Hausdorff. If $\mathscr{O}$, $\mathscr{O}'$ are two topologies on Ω, $\mathscr{O}$ finer than $\mathscr{O}'$ and $\mathscr{O}'$ Hausdorff, then also $\mathscr{O}$ is Hausdorff. In these lectures we will always consider Hausdorff spaces, even if this is not stated explicitly.

If Ω' is a non-empty subset of Ω, then the **subspace topology** on Ω' is given by $\mathscr{O}_{\Omega'} := \{\Omega' \cap O : O \in \mathscr{O}\}$. A subspace of a Hausdorff space is Hausdorff. An **isolated point** of Ω' is a point $y \in \Omega'$ for which $\{y\}$ is open in the subspace topology of Ω'. The non-isolated points of Ω' are called **accumulation points**. A point $x \in \Omega$ is the **cluster point** of the sequence $(x_n)_{n \in \mathbb{N}}$ in Ω if any neighbourhood of x contains infinitely many members of the sequence.

A subset A of a topological space Ω is called **dense** in Ω if $\overline{A} = \Omega$. A topological space Ω is called **separable** if there is a countable set $A \subset \Omega$ which is dense in Ω. A subspace of a separable metric space is separable, but a similar statement for general topological spaces is false.

A.3 Continuity

Given $(\Omega, \mathscr{O})$, $(\Omega', \mathscr{O}')$ two topological spaces, a function $f : \Omega \longrightarrow \Omega'$ is called **continuous** if the inverse image $f^{-1}(O)$ of each open set $O \in \mathscr{O}'$ is open in Ω; we sometimes say that $f : (\Omega, \mathscr{O}) \longrightarrow (\Omega', \mathscr{O}')$ is continuous. Replacing open sets by closed sets yields the same notion. The function f is **continuous at** $x \in \Omega$ if for all (open) neighbourhood V of $f(x)$ in Ω', there is U an (open) neighbourhood of x with $f(U) \subseteq V$.

For metric spaces Ω, Ω' continuity is the same as **sequential continuity**, i.e., the property that for $x_n \in \Omega$, x_n convergent to x, one has the convergence $f(x_n) \to f(x)$.

If Ω is endowed with the discrete topology then all functions $f : \Omega \longrightarrow \Omega'$ are continuous. The same is true if Ω' has the trivial topology.

A bijective continuous transformation whose inverse is also continuous is called a **homeomorphism**. The Hausdorff property is homeomorphism-invariant, whereas completeness of metric spaces is not.

A mapping $f : (\Omega, d) \longrightarrow (\Omega', d')$ between two metric spaces is called **uniformly continuous** if for each $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for all $x, y \in \Omega$. If $A \subset \Omega$ then the distance function $x \longmapsto d(x,A)$ from Ω to $\mathbb{R}$ is uniformly continuous.

A.4 Inductive and projective topologies

Let Ω_ι, $\iota \in I$ be topological spaces and $f_\iota : \Omega_\iota \longrightarrow \Omega$ for some non-empty set Ω. Define

$$\mathscr{O}_{\text{ind}} := \{A \subseteq \Omega \,:\, f_\iota^{-1}(A) \text{ for all } \iota \in I\}.$$

Then $(\Omega, \mathscr{O}_{\text{ind}})$ is a topological space. The topology $\mathscr{O}_{\text{ind}}$ is called the **inductive topology** on Ω with respect to $(f_\iota)_{\iota \in I}$, and it is the finest topology such that all the mappings f_ι become continuous. A function $g : \Omega \longrightarrow Z$, $(\Omega', \mathscr{O}')$ a topological space is continuous, if and only if all the functions $g \circ f_\iota : \Omega_\iota \longrightarrow \Omega'$, $\iota \in I$ are continuous.

The projective topology is defined on the other way round: Suppose that Ω_ι are topological spaces with $f_\iota : \Omega \longrightarrow \Omega_\iota$ mappings for some set Ω, $\iota \in I$. The **projective topology** $\mathscr{O}_{\text{proj}}$ with resect to $(f_\iota)_{\iota \in I}$ is the coarsest for which the functions f_ι become continuous. For the existence of this coarsest topology consider the family

$$\mathscr{B}_{\text{proj}} := \{f_\iota^{-1}(O_\iota) \,:\, O_\iota \in \mathscr{O} \text{ for all } \iota \in I\} \subseteq \mathfrak{P}(\Omega).$$

One can show that $\mathscr{B}_{\text{proj}}$ is a base for a topology which has the required properties.

A function $g : \Omega' \longrightarrow \Omega$, $(\Omega', \mathscr{O}')$ a topological space is continuous, if and only if all the functions $f_\iota \circ g : \Omega' \longrightarrow \Omega_\iota$, $\iota \in I$ are continuous.

An example is the *subspace topology*: If $\Omega' \subseteq \Omega$, $(\Omega, \mathscr{O})$ a topological space, then the subspace topology on Ω' is exactly the projective topology with respect to the natural imbedding $J : \Omega' \longrightarrow \Omega$.

Fig. A.1 Continuity for the inductive respectively for the projective topology

A.5 Product spaces

Let $(\Omega_\iota)_{\iota\in I}$ a non-empty family of non-empty topological spaces. The **product topology** on

$$\Omega := \prod_{\iota\in I} \Omega_\iota = \left\{ x : I \longrightarrow \bigcup \Omega_\iota \;:\; x(\iota) \in \Omega_\iota \right\}$$

is the projective topology with respect to the canonical projections $\pi_\iota : \Omega \longrightarrow \Omega_\iota$. Instead of $x(\iota)$ we usually write x_ι. A base for this topology is formed by the open rectangles

$$A_{\iota_1,\ldots,\iota_n} := \{x = (x_\iota)_{\iota\in I} \,:\, x_{\iota_i} \in O_{\iota_i} \text{ for } i = 1,\ldots,n\}$$

for $\iota_1,\ldots,\iota_n \in I$, $n \in \mathbb{N}$ and O_{ι_i} open in Ω_{ι_i}. For the product of two (or finitely many) spaces we also use the notation $\Omega \times \Omega'$ and the like. A space is Hausdorff if and only if the diagonal $\{(x,x) \,:\, x \in \Omega\}$ is closed in the product space $\Omega \times \Omega$.

If I is countable and Ω_n, $n \in I$ are all metrisable spaces then so is their product $\prod_{n\in I} \Omega_n$. The convergence in this product space is just the coordinatewise convergence. If $\Omega_\iota = \Omega$ for all $\iota \in I$, then we use the notation Ω^I for the product space.

A.6 Quotient spaces

Let $(\Omega, \mathscr{O})$ be a topological space and $\sim$ an equivalence relation on Ω, with quotient mapping

$$q : \Omega \longrightarrow \Omega/\sim,$$

sending each $x \in \Omega$ to its equivalence class. The inductive topology on $\Omega/\sim$ with respect to q is called the **quotient topology**. A set $A \subseteq \Omega/\sim$ is open in $\Omega/\sim$ if and only if $\bigcup A$, the union of the elements in A is open in Ω.

For example consider $\Omega = [0,1]$ and the equivalence relation $\sim$ whose equivalence classes are $\{0,1\}$, $\{x\}$, $x \in (0,1)$. Then the quotient $[0,1]/\sim$ is homeomorphic to the unit circle $\mathbb{T}$, under the mapping $(x \longmapsto \mathrm{e}^{2\pi \mathrm{i} \cdot x}) : [0,1]/\sim \longrightarrow \mathbb{T}$. This example is just the same as factorising $\mathbb{R}$ by $\mathbb{Z}$: take $\Omega = \mathbb{R}$ and define $x \sim y$ if $x - y \in \mathbb{Z}$. Then we have $\mathbb{R}/\mathbb{Z} = \mathbb{R}/\sim$ homeomorphic to $\mathbb{T}$.

A.7 Spaces of Continuous Functions

For Ω a topological space the set $\mathrm{C}(\Omega)$ or $\mathrm{C}^{\mathrm{b}}(\Omega)$ of all continuous functions respectively bounded, continuous functions $f : \Omega \longrightarrow \mathbb{K}$ with pointwise multiplication and addition is an algebra over $\mathbb{K}$ ($\mathbb{K}$ stands for $\mathbb{R}$ or C).

A sequence of bounded functions on Ω is **uniformly convergent** to $f : \Omega \longrightarrow \mathbb{K}$ if

$$\sup_{x \in \Omega} |f_n(x) - f(x)| \to 0, \quad \text{for } n \to \infty.$$

If each f_n is continuous and the sequence $(f_n)_n$ converges to f uniformly, the function f is continuous. The function $(f \longmapsto \|f\|_\infty) : \mathrm{C}^{\mathrm{b}}(\Omega) \longrightarrow \mathbb{R}$ is a norm and turns $\mathrm{C}^{\mathrm{b}}(\Omega)$ into a Banach space, even a Banach algebra. If (Ω, d) is a metric space, then the space $\mathrm{BUC}(\Omega, d)$ of bounded uniformly continuous functions on Ω is a closed subspace of $\mathrm{C}^{\mathrm{b}}(\Omega)$.

For general topological spaces the space $\mathrm{C}(\Omega)$ may be quite "small". For example, if Ω carries the trivial topology, the only continuous functions thereon are the constant ones. In "good" topological spaces the continuous functions **separate the points**, i.e., for every $x, y \in \Omega$ such that $x \neq y$ there is $f \in \mathrm{C}(\Omega)$ such that $f(x) \neq f(y)$. (Such spaces are necessarily Hausdorff.) Even better it is, when the continuous functions **separate closed sets**. This means that for every pair of disjoint closed subsets $A, B \subset \Omega$ there is a function $f \in \mathrm{C}^{\mathrm{b}}(\Omega)$ such that

$$0 \le f \le 1, \quad f(A) \subset \{0\}, \quad f(B) \subset \{1\}.$$

Metric spaces do have this property: Let (Ω, d) be a metric space. For each $A \subset \Omega$ the function $x \longmapsto d(x, A)$ is continuous; moreover, it is equal to zero precisely on $\overline{A}$. Hence if A, B are disjoint closed subsets of Ω then the function

$$f(x) := \frac{d(x,B)}{d(x,A) + d(x,B)} \qquad (x \in \Omega)$$

separates A from B. (Note that f is even uniformly continuous.)

A.8 Compactness

A topological space $(\Omega, \mathscr{O})$, $\mathscr{O}$ the family of open sets in Ω, is called **compact** if it is Hausdorff and every open cover of Ω has a finite subcover. This latter condition is equivalent to the **finite intersection property**: every family of closed subsets of Ω, every finite subfamily of which has non-empty intersection, has itself non-empty intersection. A subset $\Omega' \subseteq \Omega$ is compact if Ω' with the subspace topology is compact. A compact set in a Hausdorff space is closed, and a closed subset in a compact space is compact. A **relatively compact** set is set whose closure is compact.

By the Heine-Borel theorem, a subset of $\mathbb{R}^d$ is compact iff it is closed and bounded.

The continuous image of a compact space is compact, if it is Hausdorff. Moreover, if Ω is compact and Ω' is Hausdorff, a mapping $\varphi : \Omega \longrightarrow \Omega'$ is already a homeomorphism if it is continuous and bijective. In particular, if Ω is compact for some topology $\mathscr{O}$ and if $\mathscr{O}'$ is another topology on Ω, coarser than $\mathscr{O}$ but still Hausdorff, then $\mathscr{O} = \mathscr{O}'$.

Theorem A.2 (Tychonov). *Suppose $(\Omega_\iota)_{\iota \in I}$is a family of non-empty topological spaces. Then the product space $\Omega = \prod_{\iota \in I} \Omega_\iota$ is compact if and only if each Ω_ι, $\iota \in I$ is compact.*

A metric space Ω is compact if and only if it is **sequentially compact**, that is, each sequence $(x_n) \subseteq \Omega$ has a convergent subsequence. A compact metric space is complete, separable and has a countable base, and every continuous function on it is bounded and uniformly continuous.

A Hausdorff topological space $(\Omega, \mathscr{O})$ is called **locally compact** if each of its points has a compact neighbourhood. It follows then that the topology has a base consisting of relatively compact, open sets. A compact space is (trivially) locally compact. The **support** of a function $f : \Omega \longrightarrow \mathrm{C}$ is the set

$$\operatorname{supp} f := \overline{\{x \in \Omega \,:\, f(x) \neq 0\}}.$$

The set of all continuous functions with compact support is a vector space, denoted by $\mathrm{C_c}(\Omega)$. On locally compact spaces, continuous functions separate closed from compact subsets:

Lemma A.3 (Urysohn's lemma). *Let $(\Omega, \mathscr{O})$ be a locally compact space, and let A, B disjoint closed subsets of Ω, with B compact. Then there exists a continuous function $f : \Omega \longrightarrow [0,1]$ such that f has compact support, $f(A) \subseteq \{0\}$, and $f(B) \subseteq \{1\}$.*

Let $\Omega = K$ be compact. Then $\mathrm{C}(K) = \mathrm{C_c}(K) = \mathrm{C^b}(K)$ is a Banach algebra with respect to the uniform norm. Urysohn's Lemma shows that $\mathrm{C}(K)$ separates closed sets.

A.9 Metrisability

There are various sufficient conditions for the existence of a metric on a topological space that induces the given topology. The most convenient for us is the following: Suppose that $(K,\mathscr{O})$ is a compact space and there is a countable family of functions f_n, $n \in \mathbb{N}$ that separates the points of K. Then the function

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f_n(x) - f_n(y)|}{1 + |f_n(x) - f_n(y)|}$$

is continuous on $K \times K$ and it is a metric. Thus the topology $\mathscr{O}_d$ induced by d is coarser than the original one. This means that $\mathrm{Id} : (K,\mathscr{O}) \longrightarrow (K,\mathscr{O}_d)$ is a continuous mapping, but then because of the compactness of K it is also a homeomorphism, hence $\mathscr{O}_d = \mathscr{O}$.

More generally, a compact space is metrisable if and only if it has a countable base if and only if $\mathrm{C}(K)$ is separable.

A.10 Category

A subset A of a topological space Ω is called **nowhere dense** if its closure $\overline{A}$ has empty interior: $(\overline{A})^\circ = \emptyset$. A is called of **first category** in Ω if it is the union of countably many nowhere dense subsets of Ω. A is called of **second category** in Ω if it is not of first category. Countable unions of sets of first category are of first category.

One should have the picture in mind that sets of first category are small, whereas sets of second category are large. Typically one expects that "fat" sets, for example non-empty open sets are somehow large. This requirement is the defining characteristic of Baire spaces: $(\Omega,\mathscr{O})$ is called a **Baire space** if every non-empty open subset of Ω is of second category in Ω.

Theorem A.4. *Each locally compact space and each complete metric space is a Baire space.*

A countable intersection of open sets in a topological space is called a G_δ **set**; analogously, F_σ **sets** are those that can be written as countable union of closed sets. The following is an easy consequence of the definitions:

Theorem A.5. *Let Ω be a Baire-space.*

a) *An F_σ set is of first category if and only if it has empty interior.*
b) *A G_δ set is of first category if and only if it is nowhere dense.*
c) *A countable intersection of dense G_δ sets is dense.*

Note that in a metric space every closed subset A is G_δ since $A = \bigcap_{n\in\mathbb{N}} \{x : d(x,A) < 1/n\}$.

A.11 Polish spaces

A topological space Ω is called a **Polish space** if it is separable and its topology comes from some complete metric. A locally compact space is Polish if and only if it has a countable base for its topology. A compact space is Polish if and only if it is metrisable.