Numerical Analysis and Complexity of Stochastic Processes

18th Jyväskylä Summer School

Klaus Ritter

TU Darmstadt, Germany

August 2008

Outline

Outline

Stochastic Processes

Stochastic process (real-valued): a family $X = (X(t))_{t \in D}$ of random variables

$$X(t) = X(t, \cdot) : \Omega \to \mathbb{R}$$

on a probability space $(\Omega, \mathfrak{A}, P)$.

Paths (trajectories) of X: for fixed $\omega \in \Omega$

$$D \to \mathbb{R}, \qquad t \mapsto X(t, \omega).$$

In these lectures

$$D = [0,1]^d$$

with $d \in \mathbb{N}$, and X has (at least) continuous paths, i.e.,

$$\forall \omega \in \Omega : X(\cdot, \omega) \in C(D).$$

Then X defines a continuous **random function**

$$X : \Omega \to C(D), \qquad \omega \mapsto X(\cdot, \omega).$$

Terminology, if $d > 1$: X **random field**.

X **second-order process** if

$$\forall t \in D : \mathrm{E}(X^2(t)) < \infty.$$

In this case, the **mean** and the **covariance kernel**,

$$m : D \to \mathbb{R}, \qquad K : D \times D \to \mathbb{R},$$

of X are given by

$$\begin{aligned} m(t) &= \mathrm{E}(X(t)), \\ K(s,t) &= \mathrm{E}\big((X(s) - m(s)) \cdot (X(t) - m(t))\big). \end{aligned}$$

Clearly $\big(X(t_1), \dots, X(t_n)\big)$ has mean vector $(m(t_i))_{1 \leq i \leq n}$ and covariance matrix $(K(t_i, t_j))_{1 \leq i,j \leq n}$.

X **Gaussian process** if

$$(X(t_1), \ldots, X(t_n)) \quad \text{normally distributed}$$

for every $n \in \mathbb{N}$ and all $t_1, \ldots, t_n \in D$.

Measures on function spaces: Consider the sup-norm on $C(D)$ and the respective Borel-σ-algebra $\mathfrak{B}$. Then

$$P_X(B) = P(\{X \in B\}), \qquad B \in \mathfrak{B},$$

is well-defined. Clearly P_X is a probability measure on $(C(D), \mathfrak{B})$, called the **distribution** of X.

Conversely, for any probability measure μ on $(C(D), \mathfrak{B})$,

$$X(t, f) = f(t)$$

defines a stochastic process on $(C(D), \mathfrak{B}, \mu)$, called the **canonical process**. Clearly $\mu_X = \mu$.

Clearly

$$P_X = P_Y \quad \Rightarrow \quad m_X = m_Y \wedge K_X = K_Y.$$

For Gaussian processes X and Y we have '$\Leftrightarrow$'.

Outline

Brownian Motion

In the sequel

$$D = [0, 1].$$

X **one-dimensional Brownian motion** if

(i) $X(0) = 0$,

(ii) $X(t_1) - X(t_0), \dots, X(t_n) - X(t_{n-1})$ independent for every $n \in \mathbb{N}$ and all $0 = t_0 < \cdots < t_n \leq 1$,

(iii) $X(t) - X(s) \sim N(0, t - s)$ for $0 \leq s < t$,

(iv) X has continuous paths.

Existence: Let $Z_1, \ldots, Z_n$ iid., $\mathrm{E}(Z_1) = 0$, $\mathrm{Var}(Z_1) = 1$,

$$X_n(i/n) = \sum_{j=1}^{i} \frac{1}{\sqrt{n}} \cdot Z_j, \qquad i = 0, \ldots, n,$$

and use piecewise linear interpolation. Then

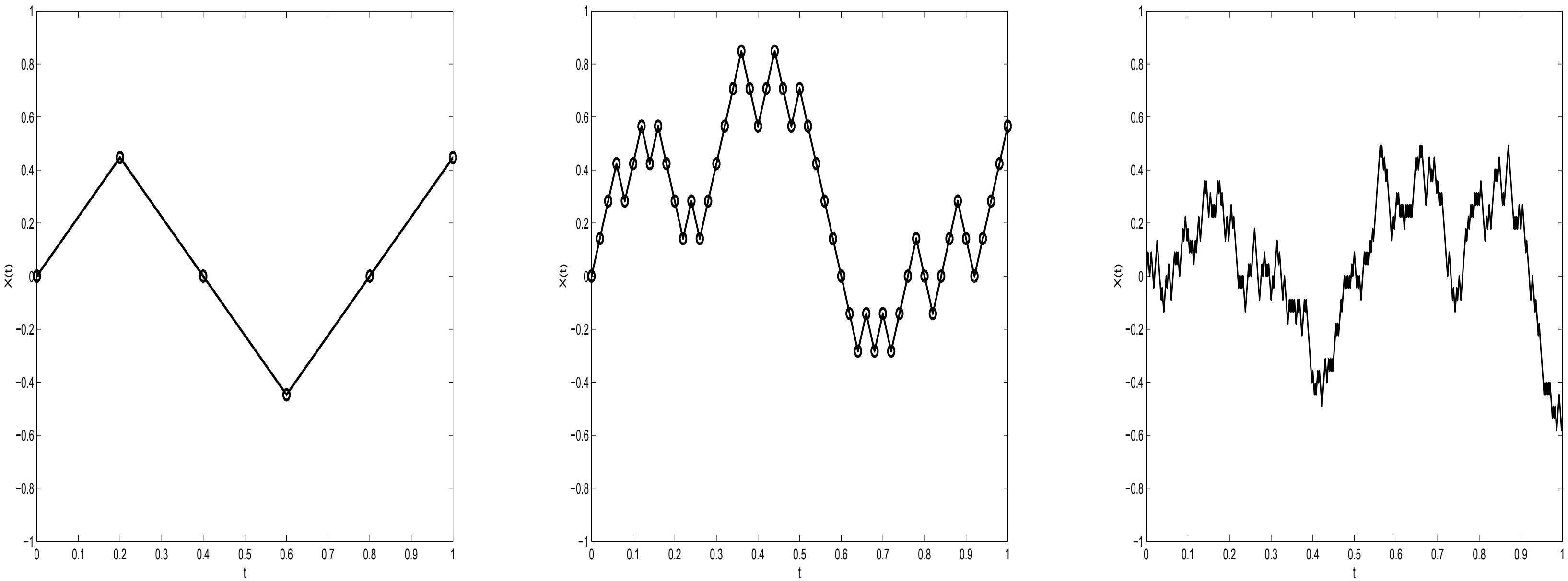
$$P_{X_n} \xrightarrow[(n\to\infty)]{\text{weakly}} w.$$

The canonical process on $(C(D), \mathfrak{B}, w)$ is a Brownian motion, and w is called the **Wiener measure**. Compare: central limit theorem.

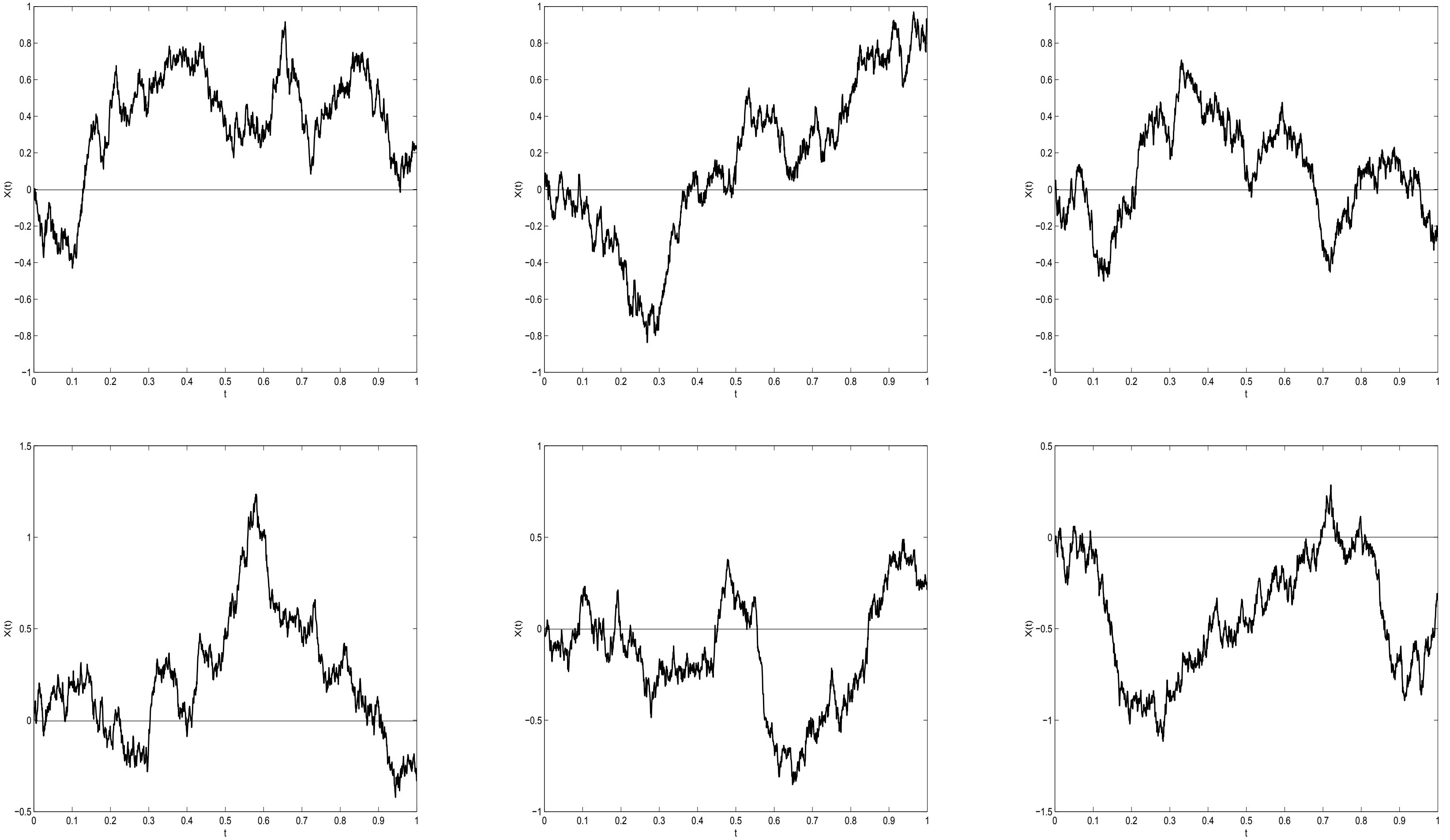
Simulation: Use random numbers to simulate realizations of $Z_1, \ldots, Z_n$.

Simulations with

$$P(Z_1 = \pm 1) = 1/2 \qquad \text{and} \qquad n = 5,\ 50,\ 500.$$



Simulations with $Z_1 \sim N(0,1)$ and $n = 800$.



Uniqueness: $P_X = w$ for every Brownian motion X.

Applications: physics, finance, analysis, ...

Lemma Every Brownian motion is Gaussian with zero mean and covariance kernel

$$K(s,t) = \min(s,t).$$

Proof By (i)–(iii), X is Gaussian. Moreover,

$$m(t) = \mathrm{E}(X(t)) = \mathrm{E}(X(t) - X(0)) = 0$$

and

$$K(t,t) = \mathrm{E}(X^2(t)) = \mathrm{E}((X(t) - X(0))^2) = t.$$

If $s \leq t$ then

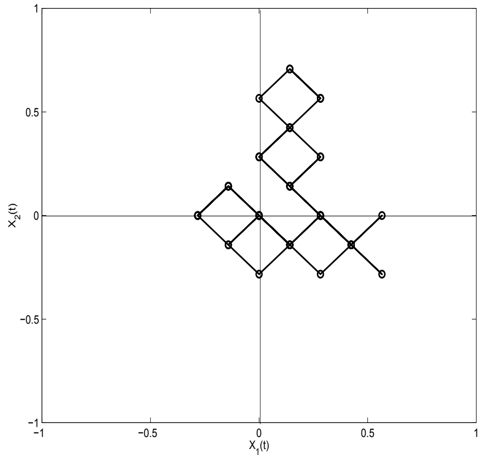
$$\begin{aligned} K(s,t) &= \mathrm{E}(X^2(s)) + \mathrm{E}\big(X(s) \cdot (X(t) - X(s))\big) \\ &= s + \mathrm{E}(X(s) - X(0)) \cdot \mathrm{E}(X(t) - X(s)) \\ &= s. \end{aligned}$$

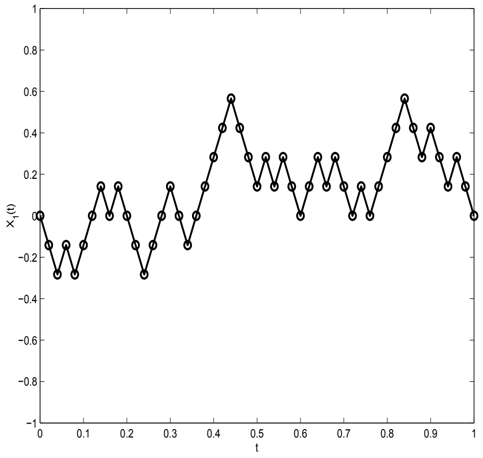
k-dimensional Brownian motion

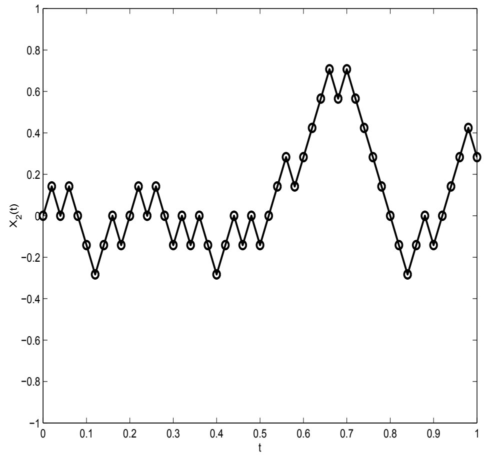
$$X(t) = \left(X^{(1)}(t), \dots, X^{(k)}(t)\right)$$

with independent one-dimensional Brownian motions $X^{(1)}, \dots, X^{(k)}$.

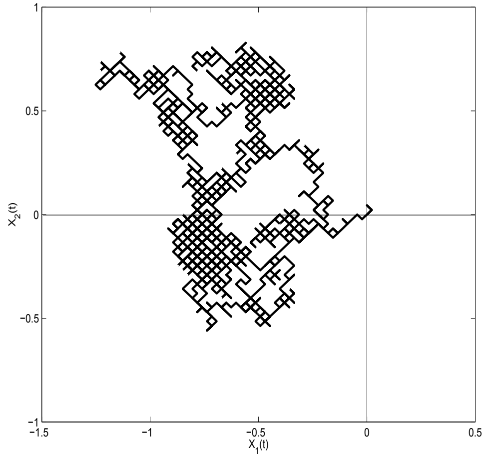
Simulation with $P(Z_1 = \pm 1) = 1/2$ and $n = 50$ for both components.



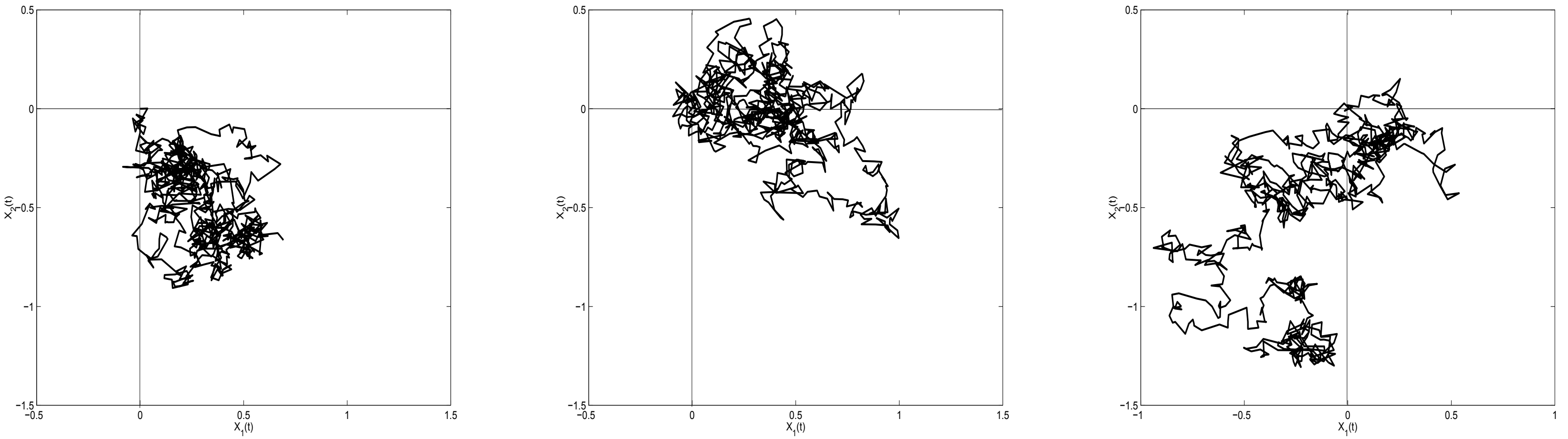




Simulation with $P(Z_1 = \pm 1) = 1/2$ and $n = 2000$ for both components.



Simulations with $Z_1 \sim N(0,1)$ and $n = 800$ for both components.



Literature

- **Stochastic processes, measures on function spaces**

V. I. Bogachev (1998), Gaussian Measures, Amer. Math. Soc., Providence.

I. Karatzas, S. E. Shreve (1999), Brownian Motion and Stochastic Calculus, Springer, New York.

M. A. Lifshits (1995), Gaussian Random Functions, Kluwer, Dordrecht.

L. Partsch (1984), Vorlesungen zum eindimensionalen Wiener Prozeß, Teubner, Leipzig.

N. N. Vakhania, V. I. Tarieladze, S. A. Chobanyan (1987), Probability Distributions on Banach Spaces, Reidel, Dordrecht.

- **Random numbers**

J. E. Gentle (1998), Random Number Generation and Monte Carlo Methods, Springer-Verlag, New York.

P. L'Ecuyer (1998), Random number generation, in: Handbook on Simulation, J. Banks (ed.), Wiley, New York.

D. E. Knuth (1998), The Art of Computer Programming, Vol. 2, Seminumerical Algorithms, Addison-Wesley, New York.

G. Marsaglia (1995), The Marsaglia random number CDROM, including the DIEHARD battery of tests of randomness, Dep. of Statistics, Florida State Univ., Tallahassee.

H. Niederreiter (1992), Random Number Generation and Quasi-Monte Carlo Methods, SIAM, Philadelphia.

Outline

Reconstruction of a Brownian Motion X

Given $n \in \mathbb{N}$, choose

knots	$0 < t_1 < \cdots < t_n \leq 1,$
functions	$a_1, \ldots, a_n \in C([0,1]),$

such that $X - X_n$ is 'small', where

$$X_n(t) = \sum_{i=1}^{n} X(t_i) \cdot a_i(t).$$

Here **mean-square L_2-distance**. For every $\omega \in \Omega$

$$\|X(\cdot,\omega) - X_n(\cdot,\omega)\|_2 = \left(\int_0^1 |X(t,\omega) - X_n(t,\omega)|^2 \, dt \right)^{1/2}.$$

The **(average) error** of X_n is

$$e(X_n) = \left(\mathrm{E}(\|X - X_n\|_2^2) \right)^{1/2}.$$

***n*-th minimal error** (minimization w.r.t. the a_i's and t_i's)

$$e(n) = \inf_{X_n} e(X_n).$$

Minimization of $e(X_n)$ for fixed knots t_i: Recall

$$e^2(X_n) = \int_D \mathrm{E}(X(t) - X_n(t))^2 \, dt.$$

Put

$$\Sigma = (K(t_i, t_j))_{1 \le i,j \le n}, \; b(t) = (K(t, t_i))_{1 \le i \le n}, \; a(t) = (a_i(t))_{1 \le i \le n}$$

to obtain

$$\begin{aligned}
\mathrm{E}(X(t) - X_n(t))^2 &= \mathrm{E}(X^2(t)) - 2\,\mathrm{E}(X(t) \cdot X_n(t)) + \mathrm{E}(X_n^2(t)) \\
&= K(t,t) - 2 \sum_{i=1}^{n} a_i(t) K(t, t_i) + \sum_{i,j=1}^{n} a_i(t) a_j(t) K(t_i, t_j) \\
&= K(t,t) - 2\, a(t)^\top b(t) + a(t)^\top \Sigma\, a(t).
\end{aligned}$$

For every $t \in D$ this is a quadratic functional w.r.t $a(t)$. Minimizer,

$$a^*(t) = \Sigma^{-1} b(t).$$

Conclusion: With $a^*(t) = \Sigma^{-1} b(t)$

$$X_n^*(t) = \sum_{i=1}^{n} X(t_i) \cdot a_i^*(t)$$

has minimal error among all X_n that use the knots t_i. Furthermore,

$$e^2(X_n^*) = \int_D \Big(K(t,t) - b(t)^\top \Sigma^{-1} \, b(t) \Big) \, dt.$$

Valid for every second-order process X (if $\det \Sigma \neq 0$)!

Explicit formulas in the Brownian motion case: Here

$$\Sigma = \begin{pmatrix} t_1 & t_1 & \dots & t_1 \\ t_1 & t_2 & \dots & t_2 \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \dots & t_n \end{pmatrix} = B\,C\,B^\top,$$

where, with $t_0 = 0$,

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 1 & \dots & \dots & 1 \end{pmatrix}, \qquad C = \mathrm{diag}(t_i - t_{i-1})_{1 \le i \le n}.$$

It follows that X_n^* is the piecewise linear interpolation of

$$0, X(t_1), \dots, X(t_n), X(t_n)$$

at the knots $0, t_1, \dots, t_n, 1$.

Moreover,

$$e^2(X_n^*) = \frac{1}{6}\sum_{i=1}^{n}(t_i - t_{i-1})^2 + \frac{1}{2}(1 - t_n)^2.$$

Note: X_n^* is ‘local’, which reflects the Markov property of X.

Minimization of $e(X_n^*)$ with respect to the knots t_i:
Easy to see: $e(X_n^*)$ is minimal iff

$$t_i^* = \frac{3i}{3n+1}.$$

Theorem (Suldin 1960) *For piecewise linear interpolation X_n^* with knots t_i^**

$$e(X_n^*) = e(n) = \frac{1}{\sqrt{2(3n+1)}}.$$

Used so far: $m = 0$ and $K = \min$, but not: X Gaussian.

Notation: strong equivalence

$$a(n) \approx b(n) \quad \text{if} \quad \lim_{n\to\infty} a(n)/b(n) = 1,$$

weak equivalence

$$a(n) \asymp b(n) \quad \text{if} \quad c_1\, a(n) \leq b(n) \leq c_2\, a(n)$$

for sufficiently large n with constants $c_1, c_2 > 0$.

Equidistant knots $t_i = i/n$ yield

$$e(X_n^*) = \frac{1}{\sqrt{6}} \cdot n^{-1/2} \approx e(n).$$

Simulation: Clearly

$$Y_i = X(t_i) - X(t_{i-1}) \sim N(0, t_i - t_{i-1}), \text{ independent.}$$

Use random numbers to simulate realizations of $Y_1, \dots, Y_n$.
Compare p. 3.

Reconstruction in L_p-norm: For $1 \le p \le \infty$ and $1 \le q < \infty$, error of X_n

$$e_{p,q}(X_n) = \left(\mathrm{E}(\|X - X_n\|_p^q)\right)^{1/q}$$

and n-th minimal error

$$e_{p,q}(n) = \inf_{X_n} e_{p,q}(X_n).$$

So far $p = q = 2$. We have

$$e_{p,q}(n) \asymp \begin{cases} n^{-1/2} & \text{if } p < \infty \\ n^{-1/2} \cdot \sqrt{\ln n} & \text{if } p = \infty. \end{cases}$$

In all cases $e_{p,q}(X_n^*) \asymp e_{p,q}(n)$ for piecewise linear interpolation X_n^* based on equidistant knots.

Smoothness of a Brownian motion

- in mean-square sense:

$$\sqrt{\mathrm{E}(X(s) - X(t))^2} = |s - t|^{1/2},$$

- pathwise: almost surely (Lévy's modulus of continuity)

$$\limsup_{h \to 0+} \sup_{|s-t|<h} \frac{|X(s) - X(t)|}{\sqrt{2\, h \ln 1/h}} = 1.$$

Decomposition of a Brownian motion

$X - X_n^*$ is zero mean Gaussian with covariance kernel

$$K(s,t) = \begin{cases} \dfrac{(\min(s,t) - t_{i-1})\,(t_i - \max(s,t))}{t_i - t_{i-1}} & \\ & \text{if } s,t \in [t_{i-1}, t_i] \\ \min(s,t) - t_n & \text{if } s,t \in [t_n, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, X_n^* and $X - X_n^*$ are independent.

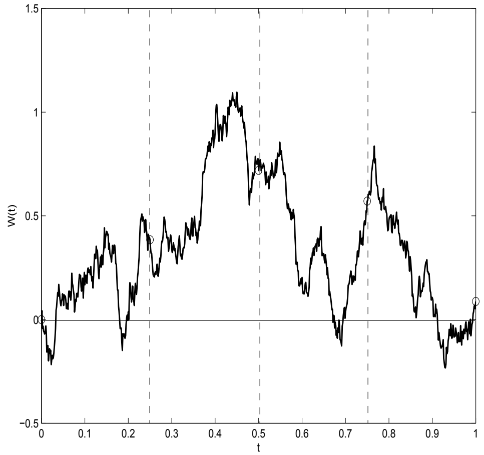
A Gaussian process on $[u, v]$ with covariance kernel

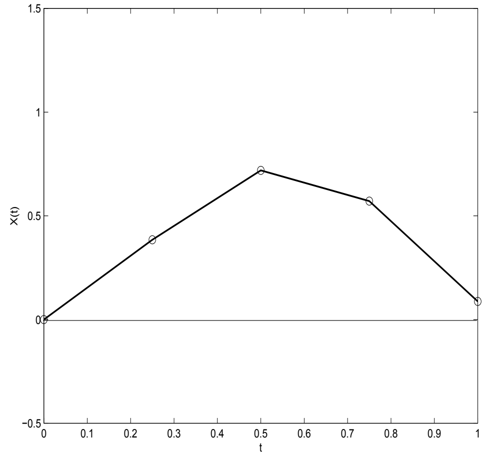
$$K(s,t) = \frac{(\min(s,t) - u)\,(v - \max(s,t))}{v - u}$$

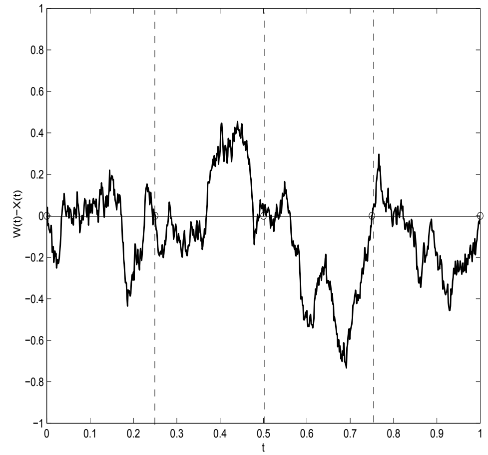
is called a **Brownian bridge**.

Simulation of corresponding trajectories of

$$X, \qquad X_n^*, \qquad X - X_n^*.$$





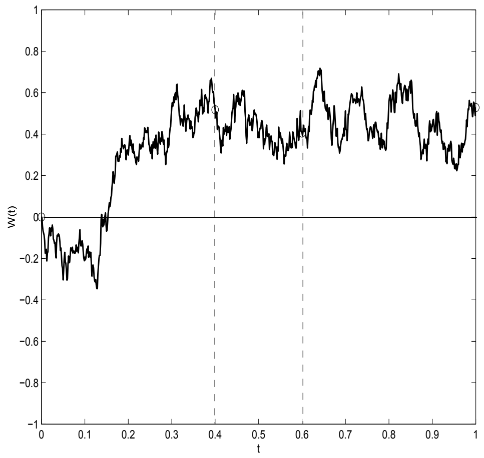


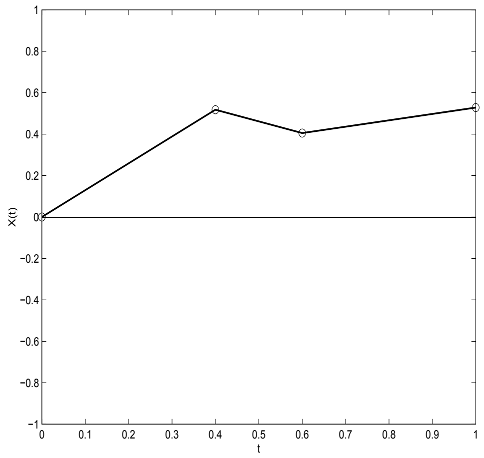
Here $n = 4$ and

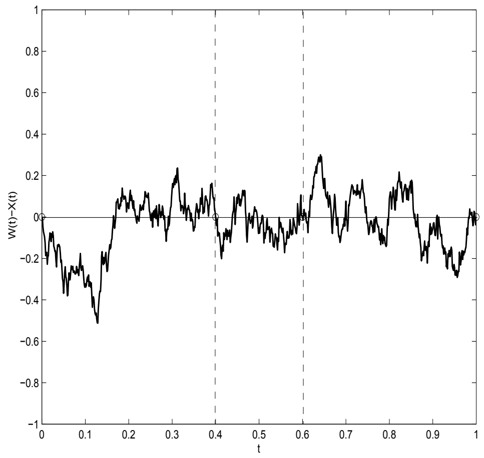
$$t_1 = 1/4, \quad t_2 = 1/2, \quad t_3 = 3/4, \quad t_4 = 1.$$

Simulation of corresponding trajectories of

$$X, \qquad X_n^*, \qquad X - X_n^*.$$





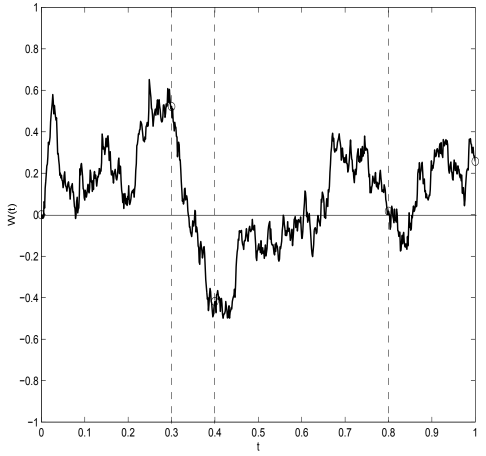


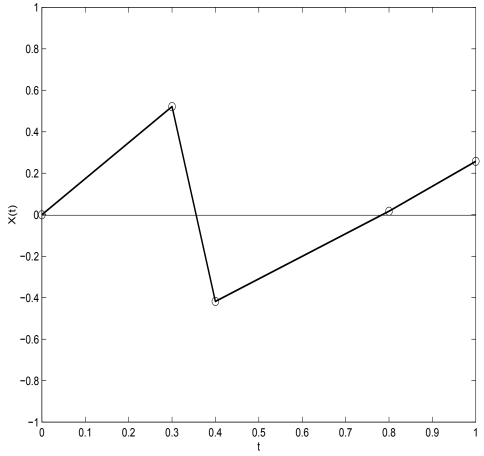
Here $n = 3$ and

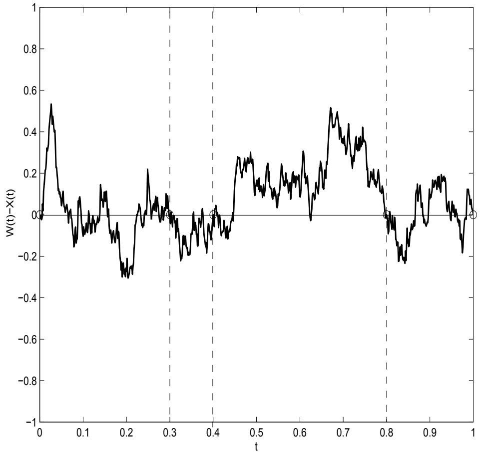
$$t_1 = 0.4, \quad t_2 = 0.6, \quad t_3 = 1.$$

Simulation of corresponding trajectories of

$$X, \qquad X_n^*, \qquad X - X_n^*.$$







Here $n = 4$ and

$$t_1 = 0.3, \quad t_2 = 0.4, \quad t_3 = 0.8, \quad t_4 = 1.$$

Literature

D. Lee (1986), Approximation of linear operators on a Wiener space, Rocky Mt. J. Math. **16**, 641–659.

K. Ritter (2000), Average-Case Analysis of Numerical Problems, Lect. Notes in Math. **1733**, Springer, Berlin.

A. V. Suldin (1959, 1960), Wiener measure and its applications to approximation methods I, II (in Russian), Isz. Vyssh. Ucheb. Zaved. Mat. **13**, 145–158, **18**, 165–179.