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Repetition Sheet Analysis I (engl.)
Winter Term 2009/10

(R.1)
Compute the supremum, infimum, maximum, minimum where they exist of the following sets

$$A = \{\frac{1}{x^2+1} \;/\; x \in \mathbb{R}\}, \qquad B = \{(-1)^n \cdot \frac{n}{n+1} \;/\; n \in \mathbb{N}\}.$$

(R.2)
Which of the following sequences are convergent? If a sequence is not convergent find the $\limsup$, $\liminf$.

$$a_n = \frac{n^8}{1+5^n}, \quad b_n = \max\{\frac{1}{n}, (-1)^n\}, \quad c_0 = 1,\ c_{n+1} = \frac{c_n}{1+c_n} \quad n \in \mathbb{N}.$$

(R.3)
Which of the following series are convergent?

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}, \quad \sum_{n=1}^{\infty} \frac{3^n}{n!}, \qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2-n}, \quad \sum_{n=1}^{\infty} (-1)^n \log(\frac{1}{n})$$

(R.4)
Let a sequence $(a_n)_{n\in\mathbb{N}}$ and assume that there is some $r > 0$ s.t. the series $\sum_{n=0}^{\infty} a_n \cdot r^n$ is convergent but not absolutely convergent. Prove that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ equals to r.

(R.5)

(a) Decide whether the following functions are continuous.

(i) $f : [0,\infty) \to \mathbb{R},\ f(x) = \sqrt{x}$,

(ii) $g : \mathbb{R} \to \mathbb{R},\ g(x) = \dfrac{\sqrt{\exp(3x^2 - 7x) \cdot (x^2+1)}}{x^4+3}$.

(b) For each point x on the surface of the Earth we denote by $T(x)$ the temperature at x at a given (fixed) point in time. We assume that T is a continuous function in x. For every point x on the surface of the Earth we define the *antipodal point* of x as the (unique) point on the surface of the earth which lies on the line which passes through x and the center of the earth. We denote the antipodal point of x by x_A. You may take for granted that the function $x \mapsto x_A$ is continuous. Prove that there exists a point x such that $T(x) = T(x_A)$ i.e., assuming that the function of the temperature is continuous there is a point on the surface of earth which has the same temperature with its antipodal point.

(R.6)
Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \dfrac{1}{q}, & x = \frac{p}{q} \in \mathbb{Q} \text{ with } \frac{p}{q} \text{ irreducible and } q > 0, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous *exactly* at all *irrational points* x_0. [1]

(R.7)

1. Let $D \subseteq \mathbb{R}$ be an open set and $f : D \to \mathbb{R}$ be a function which is continuous at $x_0 \in D$ and differentiable on $D \setminus \{x_0\}$. Use the mean value theorem to prove that if $\lim_{x \to x_0} f'(x) = a$ then then function f is differentiable at x_0 and $f'(x_0) = a$. Is it true that if the limit $\lim_{x \to x_0} f'(x)$ does not exist then f is not differentiable at x_0?
2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 \cdot \cos(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ is differentiable at all $x \in \mathbb{R}$.

(R.8)
Let the function $f : \mathbb{R} \to \mathbb{R}$ which is defined by $f(x) = \frac{\sin(x^2)}{x}$ if $x \neq 0$ and $f(0) = 0$. Prove that f is infinite many times differentiable and compute the values $f^{(n)}(0)$ for all $n \in \mathbb{N}$.

(R.9)
Prove that the condition of monotonicity is essential in the Theorem of Dini (see T14.2) i.e., find a compact set $K \subseteq \mathbb{R}$ and continuous functions $f_n : K \to \mathbb{R}$, $n \in \mathbb{N}$, such that the sequence $(f_n)_{n \in \mathbb{N}}$ is pointwise convergent to some continuous function f but not uniformly convergent to f.

(R.10)
Give the example of an integrable function $f : [-1, 1] \to \mathbb{R}$ for which the function $F(x) = \int_{-1}^{x} f(t)dt$, $x \in [-1, 1]$ is not differentiable.

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[1] The term " $\frac{p}{q}$ is irreducible" means that the only common divisor of the integers p and q is 1. This is to make the function f well defined. For example for $x_0 = \frac{3}{5} = \frac{6}{10} = \frac{12}{20}$ the assigned value is $\frac{1}{5}$ i.e, $f(x_0) = \frac{1}{5}$. Without the rule "$\frac{p}{q}$ is irreducible" the numbers $10, 20, \ldots$ could also be the values of $f(x_0)$ and so f would not be well defined. The condition "$q > 0$" is also needed to make f well defined. For example $-\frac{2}{3} = \frac{-2}{3} = \frac{2}{-3}$. Without the rule that the denominator must be positive the numbers $\frac{1}{3}$ and $\frac{1}{-3}$ could both be the values of $f(-\frac{2}{3})$. But with this rule we know that $f(-\frac{2}{3}) = f(\frac{-2}{3}) = \frac{1}{3}$.