Fachbereich Mathematik
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14th Homework Sheet Analysis I (engl.)

Winter Term 2009/10

(H14.1)

1. Define $f_n(x) = n\sin(\frac{x}{n})$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $g_n = \frac{1}{n}\sin(nx)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. Prove that the sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ converge pointwise to some differentiable functions f and g respectively. Check also if $(f_n')_{n\in\mathbb{N}}$ converges pointwise to f' and if $(g_n')_{n\in\mathbb{N}}$ converges pointwise to g'.

2. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of *continuous* functions from $\mathbb{R}$ to $\mathbb{R}$ with $\|f_n\|_\infty \leq 1$ for all $n \in \mathbb{N}$. Prove that for all $x \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot f_n(x)$ converges. Prove also that the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot f_n(x)$, $x \in \mathbb{R}$ is continuous.

(H14.2)

1. For all $n \in \mathbb{N}$ define $f_n(x) = nx(1-x^2)^n$, $x \in [0,1]$. Prove that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent and compute its limit. Is $(f_n)_{n\in\mathbb{N}}$ uniformly convergent?

2. Define the sequence of functions $(g_n)_{n\in\mathbb{N}}$ inductively as follows: $g_1(x) = x$, $x \in [0,1]$ and $g_{n+1}(x) = \sqrt{|g_n(x)|}$ for all $x \in [0,1]$. Prove that the sequence $(g_n)_{n\in\mathbb{N}}$ is pointwise convergent to some $g : [0,1] \to \mathbb{R}$ and that the sequence $(g_n)_{n\in\mathbb{N}}$ *restricted on* $[\frac{1}{2}, 1]$ is uniformly convergent to the restriction of g on $[\frac{1}{2}, 1]$. Is $(g_n)_{n\in\mathbb{N}}$ uniformly convergent on the whole interval $[0,1]$?

   Hint. Prove using induction that for all $n \in \mathbb{N}$ we have that $g_{n+1}(x) = x^{(1/2^n)}$, $x \in [0,1]$.

(H14.3)

Let the function $f(x) = \frac{1}{x^2+1}$, $x \in \mathbb{R}$.

1. Prove that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ there exists the n^{th} derive $f^{(n)}(x)$ of f at x.

   Hint. Prove that for all $n \in \mathbb{N}$ there exists a polynomial $P_n(x)$ such that $f^{(n)}(x) = \frac{P_n(x)}{(x^2+1)^{2n}}$.

2. For all $n \in \mathbb{N}$ compute the value of the n^{th} derive $f^{(n)}(0)$ of f at 0.

   Hint. Represent f as power series near 0.