Fachbereich Mathematik
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12th Homework Sheet Analysis I (engl.)
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(H12.1)

Suppose that we are given functions $f, g : \mathbb{R} \to \mathbb{R}$. Are the following statements true or false?

1. If f is differentiable at 0 then f is continuous at 0.
2. If f is continuous at 1 then f is differentiable at 1.
3. If the function $h(x) = (f(x))^2,\ x \in \mathbb{R}$ is differentiable then f is differentiable.
4. If the function $f + g$ is differentiable and f is continuous then g is differentiable.

(H12.2)

1. Find all $x \in \mathbb{R}$ for which the function f given below is differentiable at x and in this case compute the derivative $f'(x)$. Do the same for g. Also examine whether the functions f, g are Lipschitz.

$$f(x) = |x|,\ x \in \mathbb{R}, \qquad g(x) = \cos(2\pi x^3),\ x \in \mathbb{R}.$$

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that the function f' is also differentiable and that $f''(x) = 0$ for all $x \in \mathbb{R}$. Prove that f is a polynomial of degree at most 1. (*Hint.* You may use Corollary 2.5 Chap. 4).

(H12.3)

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Prove that

   (a) If $f'(x_0) = 0$ for an $x_0 \in (a, b)$, then x_0 is
   (i) a local minimum, if $f' \leq 0$ on (a, x_0) and $f' \geq 0$ on (x_0, b);
   (ii) a local maximum, if $f' \geq 0$ on (a, x_0) and $f' \leq 0$ on (x_0, b).

   (b) If $|f'(x)| \leq L$ for all $x \in [a, b]$ we have that

   $$|f(x) - f(y)| \leq L \cdot |x - y|$$

   for all $x, y \in [a, b]$. (These are (c) and (d) of Corollary 2.5 Chap. 4).

2. Define the function $f : \mathbb{R} \to \mathbb{R} : f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Prove that f is not differentiable at 0.