



10th Homework Sheet Analysis I (engl.) Winter Term 2009/10

(H10.1)

1. Which of the following sets are compact? (You have to give a proof that the set is compact or not compact).

$$(0, 1], (-\infty, 0], [0, 1] \cap \mathbb{Q}, \{\vec{x} = (x, y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}.$$

2. Determine if the follow limits exist and in this case compute those limits.

$$\lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x}, \quad \lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

(H10.2)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. Assume that the set

$$\{(x, f(x)) \in \mathbb{R}^2 / x \in \mathbb{R}\}$$

is a closed subset of \mathbb{R}^2 . Prove that the function f is continuous.

Hint. One may use sequential compactness.

(H10.3)

Let $K \subseteq \mathbb{R}^n$. Prove that K is compact if and only if for all families of sets $(F_i)_{i \in I}$ which satisfy the following properties:

1. F_i is closed for all $i \in I$,
2. if $i_1, \dots, i_m \in I$ then $F_{i_1} \cap \dots \cap F_{i_m} \cap K \neq \emptyset$,

we have that $\bigcap_{i \in I} F_i \cap K \neq \emptyset$.