Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades

TECHNISCHE UNIVERSITÄT DARMSTADT

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7th Homework Sheet Analysis I (engl.)
Winter Term 2009/10

(H7.1)

1. Prove that the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} x \cdot \exp(x), & \text{if } x \neq 0 \\ 1 & \text{, if } x = 0 \end{cases}$$

   is not continuous.

2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} \dfrac{\exp(x) - 1}{x}, & \text{if } x \neq 0 \\ 1 & \text{, if } x = 0 \end{cases}$$

   is continuous at 0.

3. Assume that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(q) = g(q)$ for all $q \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

(H7.2)

Decide whether the following statements are true or false. (In case where the statement is true you have to give a proof and in case where the statement is false you have to give a counterexample).

1. If $(q_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers which converges to some $x \in \mathbb{R}$ then x is rational.

2. If $(y_n)_{n\in\mathbb{N}}$ is a sequence of irrational numbers which converges to some $y \in \mathbb{R}$ then y is irrational.

3. For all $y \in \mathbb{R}$ there is a sequence of *rational* numbers $(q_n)_{n\in\mathbb{N}}$ such that $q_n \to y$ and $q_n > q_{n+1}$ for all $n \in \mathbb{N}$.

4. For all $y \in \mathbb{R}$ there is a sequence of *irrational* numbers $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to y$ and $y_n > y_{n+1}$ for all $n \in \mathbb{N}$.

(H7.3)

1. Assume that f_1, f_2 are continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ as follows:

   $$f(x) = \begin{cases} f_1(x), & \text{if } x \in \mathbb{Q} \\ f_2(x), & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Fix some $x_0 \in \mathbb{R}$. Prove that f is continuous at x_0 if and only if $f_1(x_0) = f_2(x_0)$.

2. Infer that the *Dirichlet function* $f : \mathbb{R} \to \mathbb{R}$:

   $$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \notin \mathbb{Q} \end{cases}$$

   is not continuous at any point $x \in \mathbb{R}$.

3. Let f, g be continuous functions from $[0, 1]$ to $\mathbb{R}$. Assume that $f(0) \geq g(0)$ and that $f(1) \leq g(1)$. Prove that there is some $\xi \in [0, 1]$ such that $f(\xi) = g(\xi)$.