Fachbereich Mathematik
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5th Homework Sheet Analysis I (engl.)

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(H5.1)

1. Which of the following series are convergent?

   (a) $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$,

   (b) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$,

   (c) $\sum_{n=1}^{\infty} \frac{3n^2+5}{n^4+5n^3+1}$.

2. Using the Ratio Test prove that for every $a \in [0, \frac{1}{e})$ the following series converges $\sum_{n=1}^{\infty} \frac{(a \cdot n)^n}{n!}$.

(H5.2)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of *non-negative* real numbers which is convergent to 0. Prove that there is a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ such that the series $\sum_{n=1}^{\infty} a_{k_n}$ is convergent.

Hint: For every $n \in \mathbb{N}$ apply the definition of convergence taking $\varepsilon = \frac{1}{2^n}$.

(H5.3)

1. Define the sequence $a_n = \begin{cases} \dfrac{1}{\sqrt{n}}, & \text{if } n = m^4 \text{ for some } k \in \mathbb{N}, \\ \dfrac{1}{n^2}, & \text{otherwise} \end{cases}$

   Prove that the series $\sum_{n=1}^{\infty} a_n$ converges.

   Remark: This exercise provides the example of a sequence $(a_n)_{n\in\mathbb{N}}$ for which the series $\sum_{n=1}^{\infty} a_n$ is convergent and yet it is not true that for some $M > 0$ we have that $|a_n| \leq \frac{M}{n}$ for all $n \in \mathbb{N}$.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a *decreasing* sequence of *non-negative* real numbers. Assume that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $n \cdot a_n \xrightarrow{n\to\infty} 0$.

Hint: Use the Cauchy's Convergence Criterion.