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2009-11-05

4th Homework Sheet Analysis I (engl.)
Winter Term 2009/10

(H4.1)

(a) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$ and $a \in \mathbb{C}$. Let us repeat the following. The sequence $(a_n)_{n\in\mathbb{N}}$ is *convergent* to a exactly when

$$(\forall\varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)[\ |a_n - a| < \varepsilon\ ],$$

therefore the sequence $(a_n)_{n\in\mathbb{N}}$ is **not** convergent to a exactly when

$$(\exists\varepsilon_0 > 0)(\forall N \in \mathbb{N})(\exists n \geq N)[\ |a_n - a| \geq \varepsilon_0\ ].$$

Assume now that the sequence $(a_n)_{n\in\mathbb{N}}$ is **not** convergent a. Prove that there is some $\varepsilon_0 > 0$ and natural numbers $n_1 < n_2 < \ldots < n_k < \ldots$ such that $|a_{n_k} - a| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

(b) Prove or reject the following statements:

(i) $(a_n)_{n\in\mathbb{N}}$ is convergent $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ has exactly one cluster point.

ii) $(a_n)_{n\in\mathbb{N}}$ has exactly one cluster point $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is convergent.

(iii) $(a_n)_{n\in\mathbb{N}}$ has exactly one cluster point and is bounded $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is convergent.

(iv) $(a_n)_{n\in\mathbb{N}}$ is bounded $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ has at most two cluster points.

(H4.2)

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be bounded sequences in $\mathbb{R}$ such that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$.

(a) Prove that

$$\limsup_{n\to\infty}\,(a_n \cdot b_n) \leq \limsup_{n\to\infty}\ a_n \cdot \limsup_{n\to\infty}\ b_n.$$

Hint: follow the proof of G4.3-(a).

(b) Assume moreover that *either* $(a_n)_{n\in\mathbb{N}}$ is convergent to some $a \in \mathbb{R}$ *or* $(b_n)_{n\in\mathbb{N}}$ is convergent to some $b \in \mathbb{R}$. Prove that we have equality in (a) i.e.,

$$\limsup_{n\to\infty} (a_n \cdot b_n) = \limsup_{n\to\infty} \ a_n \cdot \limsup_{n\to\infty} \ b_n.$$

Remark. It is not true that in (a) we have equality in general. To see this put $a_n := 1 + (-1)^n$ and $b_n := 1 + (-1)^{n+1}$, $n \in \mathbb{N}$. Then both $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded and we have that $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Moreover

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 2,$$

but on the other hand

$$a_n \cdot b_n = 1 + (-1)^{n+1} + (-1)^n + (-1)^{2n+1} = 1 + 0 - 1 = 0,$$

for all $n \in \mathbb{N}$. Thus

$$\limsup_{n\to\infty}(a_n \cdot b_n) = 0 < 2 = \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n.$$

(H4.3)

(a) Check whether the following series are convergent:

$$\text{(i)} \ \sum_{n=1}^{\infty}((-1)^n \cdot n), \qquad \text{(ii)} \ \sum_{n=1}^{\infty} \frac{(3^{n+1})^2}{17 \cdot 2^{3n}}.$$

(b) Calculate the sums of the following series:

$$\text{(i)} \ \sum_{n=1}^{\infty}\Big(\frac{1}{n} - \frac{1}{n+1}\Big), \qquad \text{(ii)} \ \sum_{n=0}^{\infty}\Big[\sum_{k=0}^{n} \binom{n}{k} \Big(\frac{1}{2}\Big)^{n+k}\Big].$$