Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades

TECHNISCHE UNIVERSITÄT DARMSTADT

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3rd Homework sheet Analysis I (engl.)
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(H3.1)

1. Prove using the definition that the following sequences are convergent to 0.

   (i) $a_n = \begin{cases} \dfrac{1}{2^n}, & \text{if } n \in \mathbb{N} \text{ is even,} \\[2ex] \dfrac{1}{n^2}, & \text{if } n \in \mathbb{N} \text{ is odd.} \end{cases}$

   (ii) $b_n = \dfrac{1}{\sqrt{n}}$, $n \in \mathbb{N}$.

2. Check whether the following sequences are convergent and find the limits in the cases where these exist.

   (i) $c_n = \dfrac{n}{n^3 - n^2 + 3n + 1}$, $n \in \mathbb{N}$,

   (ii) $d_n = \begin{cases} 0, & \text{if } n \in \mathbb{N} \text{ is even,} \\ 1, & \text{if } n \in \mathbb{N} \text{ is odd.} \end{cases}$

(H3.2)

Let A, B be two non-empty sets of real numbers and let the function $f : A \to B$. The function f is *increasing* if it satisfies the property:

$$x \leq y \Longrightarrow f(x) \leq f(y), \quad \text{for all } x, y \in A.$$

The function f is *strictly increasing* if it satisfies the property:

$$x < y \Longrightarrow f(x) < f(y), \quad \text{for all } x, y \in A.$$

1. Prove that every strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is injective.

2. Give the example of a strictly increasing function $f : [0,1] \to [0,1]$ which is not surjective.

3. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is increasing. Define by recursion the sequence $(a_n)_{n\in\mathbb{N}}$ as follows:
$$a_1 = 1, \quad a_{n+1} = f(a_n),\ n \in \mathbb{N}.$$
(i) If $1 \leq f(1)$ prove that $a_n \leq a_{n+1}$, for all $n \in \mathbb{N}$.
(ii) Assume that $1 \leq f(1)$ and that $-2 \leq f(x) \leq 2$ for all $x \in \mathbb{R}$. Prove that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent to some $a \in [-2, 2]$.

(H3.3)

1. Suppose that $(a_n)_{n\in\mathbb{N}}$ is a sequence of real numbers which is convergent to 1. Define the sets
$$A = \{n \in \mathbb{N} \ / \ a_n < 1,0002\},\ B = \{n \in \mathbb{N} \ / \ a_n \leq 0,9999\}.$$
Examine whether A and B are finite or infinite subsets of $\mathbb{N}$. (We regard the empty set as finite).

2. Give the example of two sequences $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ of real numbers such that $b_n < c_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n$.

3. Prove that if $(z_n)_{n\in\mathbb{N}}$ is a sequence of complex numbers which is convergent to some $z \in \mathbb{C}$ then $|z_n| \to |z|$.
(Hint: Prove as in the case of the real numbers that $||z| - |w|| \leq |z - w|$ for all $z, w \in \mathbb{C}$).

4. Give the example of a sequence $(d_n)_{n\in\mathbb{N}}$ of **real** numbers such that $|d_n| \to 5$ but $(d_n)_{n\in\mathbb{N}}$ is not convergent to 5.