Fachbereich Mathematik
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14th Exercise Sheet Analysis I (engl.)

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(G14.1)

1. For all $n \in \mathbb{N}$ define $f_n(x) = \dfrac{n^2x^2}{1+n^2x^2}$, $x \in [-1,1]$. Examine whether the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent and if yes compute its limit. Is $(f_n)_{n\in\mathbb{N}}$ uniformly convergent?
2. For all $n \in \mathbb{N}$ let the function $f_n : \mathbb{R} \to \mathbb{R} : f_n(x) = \frac{\sin(\sqrt{n}\cdot x)}{n^5}$. Prove that

   (i) the series $\sum_{n=1}^{\infty} f_n(x)$ converges for all $x \in \mathbb{R}$;

   (ii) the function $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in \mathbb{R}$ is 7 times differentiable and

   (iii) the series $\sum_{n=1}^{\infty} f_n^{(8)}(x)$ does not converge uniformly.

   Hint for (iii): Use the Cauchy Criterion for uniform convergence, (Theorem 4.4 Chap. 4).

(G14.2)

Prove Weierstrass' M-test (this is Theorem 4.9 Chap.4; recall $\|g\|_\infty = \sup_{x\in D} |g(x)|$):

Let $D \subseteq \mathbb{R}$ and let $f_n : D \to \mathbb{K}$, $n \in \mathbb{N}$, be functions such that $\sum_{n=1}^{\infty} \|f_n\|_\infty$ converges. Then the series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly on D.

(G14.3)

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $\rho > 0$.

1. Prove that the term-by-term differentiated power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has the same radius of convergence ρ. (This is Lemma 4.13 Chap. 4).

   Hint. Recall from H4.2 that for all bounded sequences of real numbers $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ if $(a_n)_{n\in\mathbb{N}}$ is convergent then $\limsup_{n\in\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n)\cdot(\limsup_{n\to\infty} b_n)$.
2. Prove that the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \rho$ is infinitely many times differentiable at all $x \in (-\rho, \rho)$.