Fachbereich Mathematik
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8th Exercise Sheet Analysis I (engl.)
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(G8.1)

1. Find the boundary of the following sets:
$$[0,\sqrt{2}),\quad (0,1)\cup(\sqrt{2},\sqrt{3}),\quad [100,\infty),\quad \mathbb{Q},\quad \mathbb{R}.$$
2. Which of the previous sets are closed?
3. Find a simpler form for the following sets:
$$\bigcup_{n\in\mathbb{N}}[0,1-\frac{1}{n}],\quad \bigcup_{n\in\mathbb{N}}[\frac{1}{n},1-\frac{1}{n}),\quad \bigcap_{n\in\mathbb{N}}(0,\frac{1}{n}],\quad \bigcap_{n\in\mathbb{N}}[-3,-(1+\frac{1}{n})^n].$$

(G8.2)

1. For $r>0$ and $x\in\mathbb{R}^n$ consider the open ball
$$B_r(x)=\{y\in\mathbb{R}^n : |y-x|<r\}.$$
Prove that $B_r(x)$ is an open subset of $\mathbb{R}^n$.
2. Let $M\subseteq\mathbb{R}^n$. Prove that a point $x\in\mathbb{R}^n$ belongs to M° (i.e., x is interior point of M) if and only if there is some $r>0$ such that $B_r(x)\subseteq M$.
3. Infer that for all $M\subseteq\mathbb{R}^n$ we have that
$$M^\circ=\bigcup_{\substack{V\subseteq M\\ V \text{ open}}} V\quad.$$

(G8.3)

Prove that the only subsets of $\mathbb{R}$ which are open and closed are $\emptyset$ and $\mathbb{R}$.

Hint. You may use the following result from Real Analysis. If A,B are non-empty *closed* subsets of $\mathbb{R}$ such that $A\cap B=\emptyset$ then there exists a continuous function $f:\mathbb{R}\to[0,1]$ such that
$$x\in A\iff f(x)=0,\quad x\in B\iff f(x)=1.$$