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7th Exercise Sheet Analysis I (engl.)

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First we give the following theorem.

Theorem.

1. For any two numbers $x, y \in \mathbb{R}$ for which $x < y$ there exists a rational number q such that $x < q < y$.
2. For any two numbers $x, y \in \mathbb{R}$ for which $x < y$ there exists an irrational number z such that $x < z < y$.
3. For any $x \in \mathbb{R}$ there is a sequence $(q_n)_{n\in\mathbb{N}}$ of rational numbers such that $q_n \to x$.
4. For any $x \in \mathbb{R}$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ of irrational numbers such that $x_n \to x$.

Proof.

1. We have that $x < y$, so $y-x > 0$. By the Archimedean principle there exists $k \in \mathbb{N}$ with $1/k < y-x$. Using this k we define the set
$$A := \{n \in \mathbb{Z} : n > kx\}.$$
This set is not empty, for otherwise $\mathbb{Z}$ would be bounded from above by kx. Furthermore, A is bounded from below by kx. Thus A has a least element m. (This follows analogously to the statement (f) of Theorem 1.12 Chap. 1). We now show that $q := m/k$ has the property we are after. Firstly q is clearly rational. We also have that $m \in A$ and so $x < m/k = q$ (notice that $k > 0$).
Furthermore we get $m - 1 \notin A$ since m is the minimum of A. So $m - 1 \leq kx$, which implies that $(m - 1)/k \leq x$. In total we get
$$x < q = \frac{m}{k} = \frac{m-1}{k} + \frac{1}{k} \leq x + \frac{1}{k} < x + y - x = y,$$
which is what we wanted to prove.
2. The number $\sqrt{2}$ is irrational. Put $x' = x/\sqrt{2}$ and $y' = y/\sqrt{2}$. Since $x < y$ it follows that $x' < y'$. From (1) there is a rational number q such that $x' < q < y'$. Therefore $x/\sqrt{2} < q < y/\sqrt{2}$ i.e., $x < \sqrt{2}\cdot q < y$. Assume first that $q \neq 0$. If the number $\sqrt{2}\cdot q$ was rational then the number $\sqrt{2} = (\sqrt{2}\cdot q)\cdot \frac{1}{q}$ would be rational too (as a product of two rational numbers - see G7.2). The latter is a contradiction so the number $\sqrt{2}\cdot q$ is irrational. Hence for $z = \sqrt{2}\cdot q$ we have that $x < z < y$ and z is irrational.

If $q = 0$ (and so $x' < 0 < y'$) we can choose another rational number q' s.t. $0 < q' < y'$. So $x' < q' < y'$ and $q' \neq 0$. Repeat the previous steps with q' instead of q.

Comment. Instead of the number $\sqrt{2}$ one can choose the number e which is also irrational (see Theorem 3.7 Chap. 2). In fact any positive irrational number would get us the result.

3. Let $x \in \mathbb{R}$. From (1) we have that for all $n \in \mathbb{N}$ there exists a rational number q_n such that $x - \frac{1}{n} < q_n < x + \frac{1}{n}$. Thus we have a sequence $(q_n)_{n \in \mathbb{N}}$ s.t. $|q_n - x| < \frac{2}{n}$ for all $n \in \mathbb{N}$. It follows that $q_n \to x$.
4. Exactly as in the proof of (3). Notice though that in this case we need to use (2).

We now proceed to the exercises.

(G7.1)

1. Decide whether the following functions are continuous.
   (a) $f : \mathbb{R} \to \mathbb{R} : f(x) = \exp(x^2 + 1)$.
   (b) $g : [0, 1] \cup \{2\} \to \mathbb{R} : f(x) = x$, for $x \in [0, 1]$ and $f(2) = 100$.
2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(q) = q^2$ for all $q \in \mathbb{Q}$. Prove that $f(\sqrt{2}) = 2$.

(G7.2)

1. Decide whether the following statements are true or false.
   (a) If $p, q \in \mathbb{Q}$ then $p + q, p \cdot q \in \mathbb{Q}$.
   (b) If $x, y \in \mathbb{R}$ and $x, y \notin \mathbb{Q}$ then $x \cdot y \notin \mathbb{Q}$.
   (c) If $x, y \in \mathbb{R}$ and $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$ then $x + y \notin \mathbb{Q}$.
2. Let the function $f : \mathbb{R} \to \mathbb{R}$ which is defined as follows:

$$f(x) = \begin{cases} x \cdot (x - 1), & \text{if } x \in \mathbb{Q} \\ 0 & \text{, if } x \notin \mathbb{Q} \end{cases}$$

   Prove that the function f is continuous exactly at 0 and 1.

(G7.3)

Let $f : [0, 2] \to \mathbb{R}$ be a continuous function such that $f(0) = f(2)$. Prove that there is $x_0 \in [0, 1]$ such that $f(x_0) = f(x_0 + 1)$.

Hint. Consider the function $g : [0, 1] \to \mathbb{R} : g(x) = f(x + 1) - f(x)$.