Fachbereich Mathematik
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5th Exercise Sheet Analysis I (engl.)
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(G5.1)

Decide whether the following series converge.

1. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$,
2. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$,
3. $\sum_{n=1}^{\infty} \frac{n^3-1}{2n^4+5n+1}$,
4. $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$.

(G5.2)

Suppose that $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers. Which of the following statements are correct and which are false? (In the case where a statement is correct you have to give a proof and in the case where it is false you have to give a counterexample).

1. If the series $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If the sequence $(n^2 a_n)_{n\in\mathbb{N}}$ converges then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.
3. If $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

4. If $a_n > 0$, $b_n > 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges.

5. If $a_n > 0$, $b_n > 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum\limits_{n=1}^{\infty} a_n$ converges then $\sum\limits_{n=1}^{\infty} b_n$ converges.

(G5.3)

For a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers define its associated *sequence of arithmetic means* $(b_n)_{n\in\mathbb{N}}$ as follows

$$b_n := \frac{1}{n} \cdot \sum_{j=1}^{n} a_j = \frac{a_1 + \ldots + a_n}{n}.$$

for all $n \in \mathbb{N}$.

1. Show that if $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ then $(b_n)_{n\in\mathbb{N}}$ converges to a as well.

2. Give an example of a divergent sequence $(a_n)_{n\in\mathbb{N}}$ such that the previous sequence $(b_n)_{n\in\mathbb{N}}$ is convergent.

Hint: Notice the following: if $n \geq n_0$ then

$$\frac{(a_1 - a) + \ldots (a_n - a)}{n} = \frac{(a_1 - a) + \ldots + (a_{n_0 - 1} - a)}{n} + \frac{(a_{n_0} - a) + \ldots + (a_n - a)}{n}$$