Fachbereich Mathematik
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4th Exercise Sheet Analysis I (engl.)
Winter Term 2009/10

(G4.1)

(a) Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers such that $n_1 < n_2 < \ldots < n_k < \ldots$. Prove that $n_k \geq k$ for all $k \in \mathbb{N}$.

(b) Prove that if $a_n \overset{n\to\infty}{\longrightarrow} a$ and $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$ then $a_{n_k} \overset{k\to\infty}{\longrightarrow} a$.

(c) Find all cluster points of the following sequences:

(i) $a_n := (-1)^n \frac{1}{\sqrt{n}},\ n \in \mathbb{N}$,

(ii) $$b_n = \begin{cases} 1 + \frac{1}{2^n}, & \text{if } n \in \mathbb{N} \text{ is of the form } 3k, \\ 2 + \frac{1}{n}, & \text{if } n \in \mathbb{N} \text{ is of the form } 3k+1, \\ 2, & \text{if } n \in \mathbb{N} \text{ is of the form } 3k+2. \end{cases}$$

(G4.2)

(a) Decide whether the following series are convergent:

(i) $\sum_{n=1}^{\infty} (-1)^n$,

(ii) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$.

(b) Find the sums of the following series:

(i) $\sum_{n=0}^{\infty} \frac{5 \cdot 3^n}{4^{n+2}}$,

(ii) $\sum_{k=2}^{\infty} \frac{2}{k^2-1}$.

(G4.3)

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be bounded real sequences.

(a) Show that

$$\begin{aligned}\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n &\le \liminf_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} (a_n + b_n) \\ &\le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.\end{aligned}$$

(b) Give an example with two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ s.t. we in (a) have "$<$" in all cases.