Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades

TECHNISCHE UNIVERSITÄT DARMSTADT

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3rd Exercise Sheet Analysis I (engl.)

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(G3.1)

Decide whether the following sequences are convergent and determine the limits in case these exist.

1. $a_n := \dfrac{n^3 - 2n + 1}{5n^3 + n^2 + 1}$, $n \in \mathbb{N}_0$.
2. $b_n := \frac{(-1)^n}{n}$, $n \in \mathbb{N}_0$,
3. $c_n := \sqrt{n^2 + 2} - \sqrt{n^2 + 1}$, $n \in \mathbb{N}_0$,
4. $d_n := (1 + \frac{1}{n})^{n^2}$, $n \in \mathbb{N}$,

(G3.2)

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Decide whether the following statements are true in general. Give in each case a proof or a counterexample.

1. If $(a_n)_{n\in\mathbb{N}}$ is bounded then $a_n \cdot \frac{1}{n} \to 0$.
2. If $(a_n)_{n\in\mathbb{N}}$ is bounded and $(b_n)_{n\in\mathbb{N}}$ is convergent then $(a_n \cdot b_n)_{n\in\mathbb{N}}$ is also convergent.
3. If $(a_n)_{n\in\mathbb{N}}$ is divergent and $(b_n)_{n\in\mathbb{N}}$ is divergent then $(a_n + b_n)_{n\in\mathbb{N}}$ is divergent.
4. If $(a_n)_{n\in\mathbb{N}}$ is convergent and $(a_n + b_n)_{n\in\mathbb{N}}$ is convergent then $(b_n)_{n\in\mathbb{N}}$ is convergent.

(G3.3)

1. (i) Prove that the function $f : [0, +\infty) \to \mathbb{R} : f(x) = x^2$ is injective but not surjective.
   (ii) Prove that the function $g : \mathbb{R} \to [0, +\infty) : g(x) = x^2$ is surjective but not injective.

2. Define the sequence $(a_n)_{n\in\mathbb{N}}$ by recursion as follows: $a_1 = 1$, $a_{n+1} = 6 \cdot \dfrac{1 + a_n}{7 + a_n}$, $n \in \mathbb{N}$.

   Prove that

   (i) $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$,

   (ii) $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,

   (iii) $a_n \to 2$.