Fachbereich Mathematik
Prof. Dr. W. Trebels
Dr. V. Gregoriades

TECHNISCHE
UNIVERSITÄT
DARMSTADT

2010-02-09

15th Tutorial Analysis I (engl.)

Winter Term 2009/10

In this tutorial we deal with the concepts "countable" and "uncountable". We make the following definitions: A set X is called *countable* if $X = \emptyset$ or if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in X which is surjective, i.e., such that $X = \{a_1, a_2, a_3, \ldots\}$. A set X is called *uncountable* if it is not countable.

Some examples of countable sets are the set of the integers $\mathbb{N}$, the set of even numbers $\{0, 2, 4, \ldots\} = \{a_k \ / \ a_k = 2(k-1),\ k \in \mathbb{N}\}$ and the set of integers $\{0, \pm 1, \pm 2, \ldots\} = \{a_k \ / \ a_k = n$ if $k = 2(n-1)$ and $a_k = -n$ if $k = 2(n-1)+1$ for some $n \in \mathbb{N}\}$.

(T15.1)

Let A be a countable set, and let $B \subseteq A$ be nonempty. Prove that B is countable.

(T15.2)

We will now show that $\mathbb{Q}$ is countable. We do this in three steps:

(a) Show that it is enough to prove that the set of positive rational numbers is countable.

(b) Find a method to place all fractions p/q with $p, q \in \mathbb{N}$ in a "quadratic" grid:

$$\begin{matrix} \bullet & \bullet & \bullet & \cdots \\ \bullet & \bullet & \bullet & \cdots \\ \bullet & \bullet & \bullet & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

(c) Prove that $\mathbb{Q}$ is countable.

(T15.3)

1. Work over the proof below of the following statement.

   The set X of all sequences in $\{0,1\}$ is uncountable.

   Proof: Assume that X is countable, i.e., $X = \{f_1, f_2, f_3, \ldots\}$, where $f_j = (a_{j1}, a_{j2}, a_{j3}, \ldots)$ and $a_{jk} \in \{0,1\}$ for all $j, k \in \mathbb{N}$. We now define a sequence $(a_j)_{j\in\mathbb{N}}$ in $\{0,1\}$ by letting

   $$a_j := \begin{cases} 1, & \text{if } a_{jj} = 0, \\ 0, & \text{if } a_{jj} = 1 \end{cases}$$

   for each $j \in \mathbb{N}$. Then $(a_j)_{j\in\mathbb{N}}$ is a sequence in X, and so by assumption there is an $m_0 \in \mathbb{N}$ s.t. $(a_j)_{j\in\mathbb{N}} = f_{m_0}$. But then $a_{m_0} = a_{m_0 m_0}$, which by the construction of $(a_j)_{j\in\mathbb{N}}$ is a contradiction.

   Remark: The method of proof used above is called *Cantor's diagonal argument.*

2. Use Cantor's diagonal argument to prove that the interval $[0,1)$ (and therefore all of $\mathbb{R}$) is uncountable.