VII. Compact Operator Semigroups

Having investigated the asymptotic behavior of the Cesàro means

$$T_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i$$

and having found convergence in many cases, we are now interested in the behavior of the powers

$$T^n$$

of $T(= T_\varphi)$ themselves. The problems and methods are functional-analytic, and for a better understanding of the occurring phenomena the theory of compact operator semigroups – initiated by Glicksberg-de Leeuw [1959] and Jacobs [1956] – seems to be the appropriate framework.

Therefore, in this lecture we present a brief introduction to this field, restricting ourselves to cases which will be applied to measure-theoretical and topological dynamical systems.

In the following, a semigroup S is a set with an associative multiplication

$$(t, s) \mapsto t \cdot s.$$

However such objects become interesting (for us) only if they are endowed with some additional topological structure.

VII.1 Definition:
A semigroup S is called a *semitopological semigroup* if S is a topological space such that the multiplication is separately continuous on $S \times S$. *Compact semigroups* are semitopological semigroups which are compact.

Remark: This terminology is consistent with that of App.D, since every compact (semitopological) group has jointly continuous multiplication (see VII.D.6) and therefore is a compact topological group.

For a theory applicable to operators on Banach spaces, it is important to assume that the multiplication is only separately continuous (see B.16). But this is still enough to yield an interesting structure theorem for compact semigroups. We present this result in the commutative case and recall first that an ideal in a commutative semigroup S is a nonempty subset J such that $SJ := \{st : s \in S\} \subseteq J$.

VII.2 Theorem:
Every commutative compact semigroup S contains a unique minimal ideal K, and K is a compact group.

Proof. Choose closed ideals $J_1, \ldots, J_n$ in S. Since

$$\emptyset \neq J_1 J_2 \ldots J_n \subseteq \bigcap_{i=1}^{n} J_i,$$

we conclude that the family of closed ideals in S has the finite intersection property, and therefore the ideal

$$K := \bigcap \{J : \; J \text{ is a closed ideal}\}$$

is non-empty by the compactness of S. By the separate continuity of the multiplication, the principal ideal $Ss = sS$ generated by $s \in S$ is closed. This shows that K is contained in every ideal of S. Next we show that K is a group: $sK = K$ for every $s \in S$ since K is minimal. Hence there exists $q \in K$ such that $sq = s$. Moreover for any $r \in K$ there exists $r' \in K$ such that $r's = r$. This implies

$$rq = r'sq = r's = r,$$

i.e. q is a unit in K. Again from $sK = K$ we infer the existence of $t(= s^{-1})$ such that $st = q$. Finally, we have to show that the multiplication on a compact semigroup which is algebraically a group is already jointly continuous. As remarked above, this is a consequence of a famous theorem of Ellis (see VII.D.6). ■

By the above theorem, in every compact commutative semigroup S we have a unique idempotent q, namely the unit of K, such that

$$K = qS$$

is an ideal in S and a compact group with unit q. Now we will apply this abstract result to semigroups generated by certain operators on Banach spaces. The situations which occurred in (IV.5) and (IV.6) are the main applications we have in mind.

VII.3 Lemma:
Let $(E;T)$ be an FDS satisfying

$(*)$ $\quad \{T^n f : n \in \mathbb{N}\}$ is relatively weakly compact for every $f \in E$.

Denote by $\mathscr{S} := \overline{\{T^n : n \in \mathbb{N}\}}$ the closure of $\{T^n : n \in \mathbb{N}\}$ in $\mathscr{L}(E)$ with respect to the weak operator topology. Then $\mathscr{S}$ and its closed convex hull $\overline{\text{co}}(\mathscr{S})$ are commutative compact semigroups.

Proof. Multiplication is separately continuous for the weak operator topology (see App.B.16), hence $\{T^n : n \in \mathbb{N}\}$ is a commutative semitopological semigroup in $\mathscr{L}(E)$. It is remarkable that separate continuity is sufficient to prove that its closure is still a semigroup and even commutative. We show the second assertion while the proof of the first is left to the reader. From the separate continuity it follows that operators in $\mathscr{S}$ commute with operators in $\{T^n : n \in \mathbb{N}\}$. Now take $0 \neq R_1, R_2 \in \mathscr{S}$, $f \in E$, $f' \in E'$ and $\varepsilon > 0$. Then there exists $R \in \{T^n : n \in \mathbb{N}\}$ such that

$$|\langle (R_2 - R)f, R_1 f' \rangle| \leqslant \frac{\varepsilon}{2} \qquad \text{and}$$
$$|\langle (R_2 - R)R_1 f, f' \rangle| \leqslant \frac{\varepsilon}{2}.$$

Therefore we have

$$\begin{aligned}|\langle (R_1R_2 - R_2R_1)f, f' \rangle| &= |\langle (R_1R_2 - R_1R + RR_1 - R_2R_1)f, f' \rangle| \\ &\leqslant |\langle (R_1(R_2 - R)f, f' \rangle| + |\langle (R - R_2)R_1 f, f' \rangle| \leqslant \varepsilon,\end{aligned}$$

which implies $R_1R_2 = R_2R_1$.

Finally, the condition $(*)$ implies that $\mathscr{S}$ is compact in $\mathscr{L}_w(E)$ (see App.B.14).

Since the closed convex hull of a weakly compact set in E is still weakly compact (see App.B.6), and since the convex hull $\text{co}(\mathscr{S})$ is a commutative semigroup, the same arguments as above apply to $\overline{\text{co}}(\mathscr{S})$. ■

Now we apply (VII.2) to the semigroups $\mathscr{S}$ and $\overline{\text{co}}(\mathscr{S})$. Thereby the semigroup $\overline{\text{co}}(\mathscr{S})$ leads to the already known results of Lecture IV.

VII.4 Proposition:
Let $(E;T)$ be an FDS satisfying $(*)$. Then T is mean ergodic with corresponding projection P, and $\{P\}$ is the minimal ideal of the compact semigroup $\overline{\text{co}}\{T^n : n \in \mathbb{N}_0\}$.

In particular, $E = F \oplus F_0$

where $F := PE = \{f \in E : Tf = f\}$

and $F_0 := P^{-1}(0) = \overline{(\text{id} - T)E} = \big\{f \in E : 0 \in \overline{\text{co}}\{T^n f : n \in \mathbb{N}_0\}\big\}$.

Proof. The mean ergodicity of T follows from (IV.4.c), and $TP = PT = P$ (see IV.3.1) shows that $\{P\}$ is the minimal ideal in $\overline{\text{co}}\{T^n : n \in \mathbb{N}_0\}$. The remaining statements have already been proved in (IV.3) except the last identity which follows from (IV.4.d). ∎

Analogous reasoning applied to the semigroup

$$\mathscr{S} := \overline{\{T^n : n \in \mathbb{N}_0\}} \subseteq \mathscr{L}_w(E)$$

yields another splitting of E into T-invariant subspaces. The main point in the following theorem is the fact that we are again able to characterize these subspaces.

VII.5 Theorem:
Let $(E;T)$ be an FDS satisfying $(*)$. Then there exists a projection

$$Q \in \mathscr{S} := \overline{\{T^n : n \in \mathbb{N}_0\}}$$

such that $\mathscr{K} := Q\mathscr{S}$

is the minimal ideal of $\mathscr{S}$ and a compact group with unit Q.

In particular, $E = G \oplus G_0$

where $G := QE = \overline{\text{lin}}\Big\{f \in E : Tf = \lambda f \text{ for some } \lambda \in \mathbb{C},\ |\lambda| = 1\Big\}$

and $GF_0 := Q^{-1}(0) = \Big\{f \in E : 0 \in \overline{\{T^n f : n \in \mathbb{N}_0\}}^{\sigma(E,E')}\Big\}$.

Proof. (VII.2) and (VII.3) imply the first part of the theorem, while the splitting $E = G \oplus G_0 = QE \oplus Q^{-1}(0)$ is obvious since Q is a projection.

The characterizations of $Q^{-1}(0)$ and QE are given in three steps:

1. We show that $Q^{-1}(0) = \{f \in E : 0 \in \overline{\{T^n f : n \in \mathbb{N}_0\}}^{\sigma}\}$. Since for every $f \in E$ map $S \mapsto Sf$ is continuous from $\mathscr{L}_w(E)$ into E_σ and since Q is contained in $\mathscr{S}$, we see that $Qf = 0$ implies $0 \in \overline{\{T^n f : n \in \mathbb{N}_0\}}$. Conversely, if $0 \in \overline{\{T^n f : n \in \mathbb{N}_0\}}$, there exists an operator R in the compact semigroup $\mathscr{S}$ such that $Rf = 0$. A fortiori
$$QRf = 0 \quad \text{and} \quad Qf = R'QRf = 0$$
where R' is the inverse of QR in the group $\mathscr{K} = Q\mathscr{S}$.

2. Next we prove that

$$QE \subseteq H := \overline{\text{lin}}\{f \in E : Tf = \lambda f \text{ for some } |\lambda| = 1\}.$$

Denote by $\widehat{\mathscr{K}}$ the character group of $\mathscr{K}$ and define for every character $\gamma \in \widehat{\mathscr{K}}$ the operator P_γ

$$P_\gamma(f) := \int_{\mathscr{K}} \overline{\gamma(S)} Sf \, \mathrm{d}m(S), \quad f \in E.$$

Here, m is the normalized Haar measure on $\mathscr{K}$, and the integral is understood in the weak topology on E, i.e.

$$\langle P_\gamma(f), f' \rangle := \int_{\mathscr{K}} \overline{\gamma(S)} \langle Sf, f' \rangle \, \mathrm{d}m(S), \quad \text{for every } f' \in E'.$$

$P_\gamma(f)$ is an element of the bi-dual E'' contained in $\overline{\text{co}}\{\overline{\gamma(S)} \cdot Sf : S \in \mathscr{K}\}$. However by Krein's theorem (App.B.6) this set is $\sigma(E, E')$-compact and hence contained in E. Therefore P_γ is a well-defined bounded linear operator on E. Now take $R \in \mathscr{K}$ and observe that

$$\begin{aligned} RP_\gamma(f) &= R\Big(\int_{\mathscr{K}} \overline{\gamma(S)} Sf \, \mathrm{d}m(S)\Big) = \int_{\mathscr{K}} \overline{\gamma(S)} RSf \, \mathrm{d}m(S) \\ &= \gamma(R) \int_{\mathscr{K}} \overline{\gamma(RS)} RSf \, \mathrm{d}m(RS) = \gamma(R) P_\gamma(f) \quad \text{for every } f \in \text{E} \end{aligned}$$

i.e., $\quad RP_\gamma = P_\gamma R = \gamma(R) P_\gamma.$

For $R := TQ$ we obtain $TP_\gamma = TQP_\gamma = \gamma(TQ)P_\gamma$ and therefore $P_\gamma(H) \subseteq H$. The assertion is proved if we show that $QE \subseteq \overline{\text{lin}} \bigcup \{P_\gamma E : \gamma \in \widehat{\mathscr{K}}\}$ or equivalently that $\{P_\gamma E : \gamma \in \widehat{\mathscr{K}}\}$ is total in QE.

Take $f' \in E'$ vanishing on the above set, i.e., such that $\int_{\mathscr{K}} \overline{\gamma(S)} \langle Sf, f' \rangle \, \mathrm{d}m(S) = 0$ for all $\gamma \in \widehat{\mathscr{K}}$ and all $f \in E$. Since the mapping $S \mapsto \langle Sf, f' \rangle$ is continuous, and since the characters form a complete orthonormal basis in $L^2(\mathscr{K}, m)$ (see App.D.7) this implies that $< Sf, f' >= 0$ for all $S \in \mathscr{K}$. In particular, taking $S = Q$ we conclude that f' vanishes on QE.

3. Finally, we show that $H \subseteq QE$. This inclusion is proved if Q, the unit of $\mathscr{K}$ is the identity operator on H. Every eigenvector of T is also an eigenvector of T^n and hence an eigenvector of $R \in \mathscr{S}$. Now take $\varepsilon > 0$ and a finite set

$$\mathcal{F} := \{f_1, \ldots, f_n\}$$

of normalized eigenvectors of T (and R) with

$$Rf_i = \lambda_i f_i, \quad |\lambda_i| = 1, \ 1 \leqslant i \leqslant n.$$

By the compactness of the torus Γ we find $m \in \mathbb{N}$ such that

$$\begin{aligned} |1 - \lambda_i^m| &\leqslant \varepsilon \quad \text{and consequently} \\ \|R^m f_i - f_i\| &\leqslant \varepsilon \quad \text{simultaneously for } i = 1, \ldots, n. \end{aligned}$$

This proves that the set

$$A_{\mathcal{F},\varepsilon} := \{R \in \mathscr{K} : \|Rf - f\| \leqslant \varepsilon \text{ for } f \in \mathcal{F}\}$$

is non-empty and closed. By the compactness of $\mathscr{K}$ we conclude that $\bigcap_{\mathcal{F},\varepsilon} A_{\mathcal{F},\varepsilon} \neq \emptyset$, i.e. $\mathscr{K}$ contains an element which is the identity operator on H. Since Q is the unit of $\mathscr{K}$ it must be the identity on H.

■

The minimal ideal $\mathscr{K}$ of $\mathscr{S}$ in the above theorem may be identified with a group of operators on $H = \overline{\text{lin}}\{f \in E : Tf = \lambda f \text{ for some } |\lambda| = 1\}$ which is compact in the weak operator topology and has unit $Q = \text{id}_H$. Moreover, the weak and strong topologies coincide on the one-dimensional orbits $\mathscr{S}f$ for every eigenvector f. Therefore the group $\mathscr{K}$ is even compact for the strong operator topology. Operators for which $H = E$ (and therefore $Q = \text{id}_E$ and $\mathscr{S} = \mathscr{K}$) are of particular importance and will be called "operators with discrete spectrum". The following is an easy consequence of these considerations.

VII.6 Corollary:
For an FDS $(E;T)$ with $\|T^n\| \leqslant c$ the following properties are equivalent:

(a) T has discrete spectrum, i.e. the eigenvectors corresponding to the unimodular eigenvalues of T are total in E.
(b) $\mathscr{S} = \overline{\{T^n : n \in \mathbb{N}_0\}} \subseteq \mathscr{L}_w(E)$ is a compact group with unit id_E.
(c) $\mathscr{S} = \overline{\{T^n : n \in \mathbb{N}_0\}} \subseteq \mathscr{L}_s(E)$ is a compact group with unit id_E.

The following example is simple, but very instructive and should help to avoid pitfalls.

VII.7 Example:
Take the Hilbert $\ell^2(\mathbb{Z})$ and the shift

$$T : (x_z) \to (x_{z+1}).$$

Then $\{T^n : n \in \mathbb{Z}\}$ is a group, its closure in $\mathscr{L}_w(\ell^2(\mathbb{Z}))$ is a compact semigroup with minimal ideal $\mathscr{K} = \{0\}$.

VII.8. Programmatic remark:
The semigroups in

$$\mathscr{S} := \overline{\{T_\varphi^n : n \in \mathbb{N}_0\}}$$

in $\mathscr{L}_w(L^p(X, \Sigma, \mu))$, $1 \leqslant p < \infty$, appearing in (measure-theoretical) ergodic theory are compact and therefore yield projections P (as in VII.4) and Q (as in VII.5) such that

$$\text{id} \geqslant Q \geqslant P \geqslant \mathbf{1} \otimes \mathbf{1},$$

where the order relation for projections is defined by the inclusion of the range spaces. While we have seen in (IV.7) that "ergodicity" is characterized by $P = \mathbf{1} \otimes \mathbf{1}$ we will study in the subsequent lectures the following "extreme" cases:

Lecture VIII:	$\text{id} = Q > P = \mathbf{1} \otimes \mathbf{1}$,
Lecture IX:	$\text{id} > Q = P = \mathbf{1} \otimes \mathbf{1}$.

VII.D Discussion

VII.D.1. Semitopological semigroups:
One might expect that semigroups S – if topologized – should have jointly continuous multiplication, i.e.,

$$(t, s) \mapsto t \cdot s$$

should be continuous from $S \times S$ into S. In fact, there exists a rich theory for such objects (see Hofmann-Mostert [1966]), but the weaker requirement of separately continuous multiplication still yields interesting results as (VII.2) (see Berglund-Hofmann [1967]) and occurs in non-trivial examples:

The one point compactification $S = \mathbb{Z} \cup \{\infty\}$ of $(\mathbb{Z}, +)$ is a semitopological semigroup if $a + \infty = \infty + a = \infty$ for every $a \in S$. But the addition is not jointly continuous since

$$0 = \lim_{n\to\infty} (n + (-n)) \neq \lim_{n\to\infty} n + \lim_{n\to\infty} (-n) = \infty.$$

Obviously, the minimal ideal is $K = \{\infty\}$.

VII.D.2. Weak vs. strong operator topology on $\mathscr{L}(E)$:
In ergodic theory it is the semigroup $\{T^n : n \in \mathbb{N}_0\}$ – $T \in \mathscr{L}(E)$ and E a Banach space – which is of interest. In most cases this semigroup is algebraically isomorphic to the semigroup $\mathbb{N}_0$. But since our interest is in the asymptotic behavior of the powers T^n, we need some topology on $\mathscr{L}(E)$. If we choose the norm topology or the strong operator topology, and if $\|T^n\| \leqslant c$, then $\{T^n : n \in \mathbb{N}_0\}$ and $\overline{\{T^n : n \in \mathbb{N}_0\}}$ become topological semigroups with jointly continuous multiplication. Unfortunately, these topologies are too fine to yield convergence in many cases. In contrast, if we take the weak operator topology, then $\overline{\{T^n : n \in \mathbb{N}_0\}}$ has only separately continuous multiplication, but in many cases (see IV.5, IV.6 and VII.3) it is compact, and convergence of T^n or some subsequence will be obtained. The following example illustrates these remarks:

Take $E = \ell^2(\mathbb{Z})$ and T the shift as in (VII.7). Then T^n does not converge with respect to the strong operator topology (Proof: If $T^n f$ converges, its limit must be a T-fixed vector, hence equal to 0, but $\|f\| = \|T^n f\|$.), but for the weak operator topology we have $\lim_{n\to\infty} T^n = 0$. The fact that the multiplication is not jointly continuous for the weak operator topology may be seen from

$$0 = \lim_{n\to\infty} T^n \cdot \lim_{n\to\infty} T^{-n} \neq \lim_{n\to\infty} (T^n \cdot T^{-n}) = \mathrm{id}.$$

VII.D.3. Monothetic semigroups:
The semitopological semigroup

$$\mathscr{S} = \overline{\{T^n : n \in \mathbb{N}_0\}} \subseteq \mathscr{L}_w(E)$$

generated by some FDS $(E; T)$ contains an element whose powers are dense in $\mathscr{S}$. Such an element is called *generating*, and the semigroup is called *monothetic*. We mention the following examples of monothetic semigroups:

(i) The set $S := \{2^{-n} : n \in \mathbb{N}\}$ and its closure $\overline{S} = \{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, endowed with topology and multiplication induced by $\mathbb{R}$, are the simplest monothetic semigroups.

(ii) The unit circle Γ is a (compact) monothetic group, and every $a \in \Gamma$ which is not a root of unity is generating (see III.8.iii).

(iii) The n-torus Γ^n, $n \in \mathbb{N}$ is a (compact) monothetic group, and $a = (a_1, \dots, a_n) \in \Gamma^n$ is generating iff $\{a_1, \dots, a_n\}$ is linearly independent in the $\mathbb{Z}$-module (see App.D.8).

(iv) $S := \Gamma \cup \{\frac{n+1}{n} e^{ni} : n \in \mathbb{N}\}$, $i^2 = -1$, is compact monothetic semigroup for the topology induced by $\mathbb{C}$, the canonical multiplication on Γ,

$$\frac{n+1}{n} e^{in} \cdot \frac{m+1}{m} e^{mi} := \frac{n+m+1}{n+m} e^{(n+m)i} \quad \text{for } n, m \in \mathbb{N}$$

and $$\frac{n+1}{n} e^{in} \cdot \gamma = \gamma \cdot \frac{n+1}{n} e^{in} := \gamma \cdot e^{in} \quad n \in \mathbb{N},\ \gamma \in \Gamma.$$

The element $2e^i$ is generating (compare Hofmann-Mostert [1966], p. 72).

VII.D.4. Compact semigroups generated by operators on $L^p(X, \Sigma, \mu)$:
The operators $T_\varphi : L^p X, \Sigma, \mu) \to L^p(X, \Sigma, \mu)$ appearing in the ergodic theory of MDS's $(X, \Sigma, \mu; \varphi)$ generate compact semigroups which will be discussed now in more generality. To that purpose, consider a probability space (X, Σ, μ) and a positive operator

$$T : L^1(X, \Sigma, \mu) \to L^1(X, \Sigma, \mu)$$

satisfying $T\mathbf{1} \leqslant \mathbf{1}$ and $T'\mathbf{1} \leqslant \mathbf{1}$. By the Riesz convexity theorem (see Schaefer [1974], V.8.2) T leaves invariant every $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, and the restrictions

$$T_p : L^p(X, \Sigma, \mu) \to L^p(X, \Sigma, \mu)$$

are contractive for $1 \leqslant p \leqslant \infty$. The semigroups

$$\mathscr{S}_p := \overline{\{T_p^n : n \in \mathbb{N}_0\}}$$

in $\mathscr{L}_w(E)$ are compact for $1 \leqslant p < \infty$: if $1 < p < \infty$, argue as in (IV.5); if $p = 1$, as in (IV.6). Moreover, it follows from the denseness of $L^\infty(\mu)$ in $L^p(\mu)$ that all these semigroups are algebraically isomorphic, and that all these weak operator topologies coincide (use App.A.2). Therefore the compact semigroups generated by T in $L^p(\mu)$ for $1 \leqslant p < \infty$ will be denoted by $\mathscr{S}$.
If $L^1(\mu)$ is separable we can find a sequence $\{\chi_n : n \in \mathbb{N}\}$ of characteristic functions which is total in $L^1(\mu)$. The seminorms

$$p_{n,m} := |\langle R\chi_n, \chi_m \rangle|, \quad R \in \mathscr{L}(L^1(\mu)),$$

induce a Hausdorff topology on $\mathscr{S}$ weaker than the weak operator topology. Since $\mathscr{S}$ is compact, both topologies coincide, and therefore $\mathscr{S}$ is a compact metrizable semigroup.

VII.D.5. Operators with discrete spectrum:
Clearly, the identity on any Banach space has discrete spectrum. More interesting examples follow:

(i) Consider $E = C(\Gamma)$ and $T := T_{\varphi_a}$ for some rotation

$$\varphi_a : z \mapsto a \cdot z.$$

The functions $f_n : z \mapsto z^n$ are eigenfunctions of T for every $n \in \mathbb{Z}$ and are total in $C(\Gamma)$ by the Stone-Weierstrass theorem. Therefore, T has discrete spectrum in $C(\Gamma)$.

(ii) The operator T_{φ_a} induced on $L^p(\Gamma, \mathcal{B}, m)$, $1 \leqslant p < \infty$, has discrete spectrum since it has the same eigenfunctions as the operator in (i) and since $C(\Gamma)$ is dense in $L^p(\mu)$ for $1 \leqslant p < \infty$.

(iii) Analogous assertions are valid for all operators induced by any rotation on a compact Abelian group (choose the characters as eigenfunctions), and we will see in Lecture VIII in which sense this situation is typical for ergodic theory.

(iv) There exist operators having discrete spectrum but unbounded powers: For $n > 2$ endow $E_n := \mathbb{C}^n$ with the norm

$$\|(x_1, \dots, x_n)\| := \max\{(n+1-i)^{-1}|x_i| : 1 \leqslant i \leqslant n\}$$

and consider the rotation operators

$$S_{(n)} : E_n \to E_n : (x_1, \dots, x_n) \mapsto (x_n, x_1, \dots, x_{n-1}).$$

Every $S_{(n)}$, $n \geqslant 2$, has discrete spectrum in E_n. An easy calculation shows that $\||S_{(n)}\| \leqslant 2$ and $\sup\{\|S_{(i)}^{n-1}\| : i \geqslant 2\} \leqslant \|S_{(n)}^{n-1}\| = n$ for all $n \geqslant 2$. Now, take the ℓ^1-direct sum $E := \bigoplus_{n \geqslant 2} E_n$ and $T := \bigoplus_{n \geqslant 2} S_{(n)}$. Clearly $\|T^i\| = i + 1$ for every $i \in \mathbb{N}$, but T has discrete spectrum in E.

VII.D.6. Semitopological vs. topological groups (the Ellis Theorem): In the remark following Definition (VII.1) we stated that a semitopological group which is compact is a topological group. Usually this fact is derived from a deep theorem of Ellis [1957] but the proof of the property we needed in Lecture VII is actually quite easy – at least for metrizable groups.

Proposition: Let G be a group, $\mathcal{O}$ a metrizable, compact Hausdorff topology on G such that the mapping

$$(g, h) \mapsto gh : G \times G \to G$$

is separately continuous. Then $(G, \mathcal{O})$ is a topological group.

Proof. Suppose that the multiplication is not continuous at $(s, t) \in G \times G$. Then there exists $\varepsilon > 0$ such that for every neighbourhood U of s and V of t

$$\varepsilon \leqslant d(st, s_U t_V)$$

for some suitable $(s_U, t_V) \in U \times V$, and $d(\cdot, \cdot)$ a metric on G generating $\mathcal{O}$. Since multiplication is separately continuous there exists a neighbourhood U_0 of s and V_0 of t, such that

$$d(st, s't) \leqslant \frac{\varepsilon}{4} \quad \text{for every } s' \in U_0,$$

and

$$d(s_{U_0}t, s_{U_0}t') \leqslant \frac{\varepsilon}{4} \quad \text{for every } t' \in V_0.$$

From this we obtain the contradiction

$$\varepsilon \leqslant d(st, s_{U_0}t_{V_0} \leqslant d(st, s_{U_0}t) + d(s_{U_0}t, s_{U_0}t_{V_0}) \leqslant \frac{\varepsilon}{2}.$$

Therefore the multiplication is jointly continuous on G.

It remains to prove that the mapping $g \mapsto g^{-1}$ is continuous on G. Take, $g \in G$ and choose a sequence $(g_n)_{n \in \mathbb{N}}$ contained in G such that $\lim_{n \to \infty} g_n = g$. Since $(G, \mathcal{O})$ is compact and metrizable, the sequence (g_n^{-1}) has a convergent subsequence in G. Thus we may assume that $\lim_{n \to \infty} g_n^{-1} = h$ for some $h \in G$. From the joint continuity of the multiplication we obtain $1 = gh = hg$, thus $h = g^{-1}$, which proves the assertion. ∎