II. Dynamical Systems

Many of the answers presented in Lecture I indicate that ergodic theory deals with pairs (X, φ) where X is a set whose points represent the "states" of a physical system while φ is a mapping from X into X describing the change of states after one time unit. The first step towards a mathematical theory consists in finding out which abstract properties of the physical state spaces will be essential. It is evident that an "ergodic theory" based only on set-theoretical assumptions is of little interest. Therefore we present three different mathematical structures which can be imposed on the state space X and the mapping φ in order to yield "dynamical systems" that are interesting from the mathematical point of view. The parallel development of the corresponding three "ergodic theories" and the investigation of their mutual interaction will be one of the characteristics of the following lectures.

II.1 Definition:

(i) $(X, \Sigma, \mu; \varphi)$ is a *measure-theoretical dynamical system* (briefly: MDS) if (X, Σ, μ) is a probability space and $\varphi : X \to X$ is a bi-measure-preserving transformation.

(ii) $(X; \varphi)$ is a *topological dynamical system* (TDS) if X is a compact space and $\varphi : X \to X$ is homeomorphism.

(iii) $(E; T)$ is a *functional-analytic dynamical system* (FDS) if E is a Banach space and $T : E \to E$ is a bounded linear operator.

Remarks:

1. The term "bi-measure-preserving" for the transformation $\varphi : X \to X$ in (i) is to be understood in the following sense: There exists a subset X_0 of X with $\mu(X_0) = 1$ such that the restriction $\varphi_0 : X_0 \to X_0$ of φ is bijective, and both φ_0 and its inverse are measurable and measure-preserving for the induced σ-algebra $\Sigma_0 := \{A \cap X_0 : A \in \Sigma\}$.
2. If φ is bi-measure-preserving with respect to μ, we call μ a φ-invariant measure.
3. As we shall see in (II.4) every MDS and TDS leads to an FDS in a canonical way. Thus a theory of FDSs can be regarded as a joint generalization of the topological theory of TDSs and the probabilistic theory of MDSs. In most of the following chapters we will either start from or aim for a formulation of the main theorem(s) in the language of FDSs.
4. DDSs *("differentiable dynamical systems")* will not be investigated in these lectures (see Bowen [1975], Smale [1967], [1980]).

Before proving any results we present in this lecture the fundamental (types of) examples of dynamical systems which will frequently reappear in the ensuing text. The reader is invited to apply systematically every definition and result to at least some of these examples.

II.2. Rotations:

(i) Let $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, Σ its Borel algebra, and m the normalized Lebesgue measure on Γ. Choose an $a \in \Gamma$ and define

$$\varphi_a(z) := a \cdot z \quad \text{for all } z \in \Gamma.$$

Clearly, $(\Gamma; \varphi_a)$ is a TDS, and $(\Gamma, \mathcal{B}, m; \varphi_a)$ an MDS.

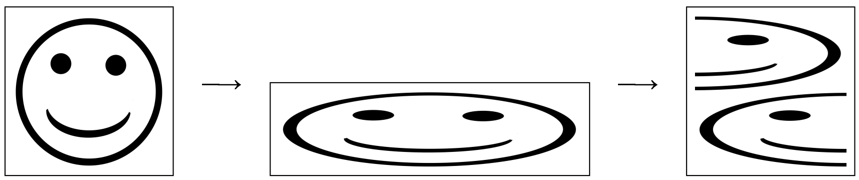
(ii) A more abstract version of the above example is the following: Take a compact group G with Borel algebra $\mathcal{B}$ and normalized Haar measure m. Choose $h \in G$ and define the (left)*rotation*

$$\varphi_h(g) := h \cdot g \quad \text{for all } g \in G.$$

Again, $(G; \varphi_h)$ is a TDS, and $(G, \mathcal{B}, m; \varphi_h)$ an MDS.

II.3. Shifts:

(i) "Dough-kneading" leads to the following bi-measure-preserving transformation



or in a more precise form: if $X := [0,1]^2$, $\mathcal{B}$ the Borel algebra on X, m the Lebesgue measure, and

$$\varphi(x,y) := \begin{cases} (2x, \frac{y}{2}) & \text{for } 0 \leqslant x \leqslant \frac{1}{2}, \\ (2x-1, \frac{(y+1)}{2} & \text{for } \frac{1}{2} < x \leqslant 1, \end{cases}$$

we obtain an MDS, but no TDS for the natural topology on X.

(ii) "Coin-throwing" may also be described in the language of dynamical systems: Assume that somebody throws a dime once a day from eternity to eternity. An adequate mathematical description of such an "experiment" is a point

$$x = (x_n)_{n \in \mathbb{Z}}$$

in the space $\widehat{X} := \{0,1\}^{\mathbb{Z}}$, which is compact for the product topology.

$$\begin{array}{rl} \text{Tomorrow, the point} & (x_n) = (\ldots\ldots, x_{-1}, x_0, x_1, x_2, \ldots) \\ & \qquad\qquad\qquad\qquad\quad \updownarrow \\ \text{will be} & (x_{n+1}) = (\ldots, x_{-1}, x_0, x_1, x_2, \ldots\ldots) \end{array}$$

where the arrow points to the current outcome of the dime-throwing experiment. Therefore, time evolution corresponds to the mapping

$$\tau : \widehat{X} \to \widehat{X}, \quad (x_n)_{n \in \mathbb{Z}} \mapsto (x_{n+1})_{n \in \mathbb{Z}}.$$

$(\widehat{X}; \tau)$ is a TDS, and τ is called the (left)*shift* on $\widehat{X}$. Let us now introduce a probability measure $\widehat{\mu}$ on $\widehat{X}$ telling which events are probable and which not. If we assume firstly, that this measure should be determined by its values on the (measurable) rectangles in $\widehat{X}$ (see A.17), and secondly, that the probability of the outcome should not change with time, we obtain that $\widehat{\mu}$ is a shift invariant probability measure on the product σ-algebra $\widehat{\Sigma}$ on $\widehat{X}$, and that $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu}; \tau)$ is an MDS.

On $\widehat{X}$ there are many τ-invariant probability measures, but in our concrete case, it is reasonable to assume further that today's outcome is independent of all the previous results, and that the two possible results of "coin throwing" have equal probabilities $p(0) = p(1) = \frac{1}{2}$. Then $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu})$ is the product space $(\{0,1\}, \mathcal{P}\{0,1\}, p)^{\mathbb{Z}}$ (see A.17).

Exercise: Show that (i) and (ii) are the "same" ! (Hint: see (VI.D.2))

(iii) Again we present an abstract version of the previous examples. Let (X, Σ, p) be a probability space, where $X := \{0, \dots, k-1\}$, $k > 1$, is finite, Σ the power set of X and $p = (p_0, \dots, p_{k-1})$ a probability measure on X.

Take $\widehat{X} = X^{\mathbb{Z}}$, the product σ-algebra $\widehat{\Sigma}$ on X, the product measure $\widehat{\mu}$ and the shift τ on $\widehat{X}$. Then we obtain an MDS $(\widehat{X}, \widehat{\Sigma}, \widehat{\mu}; \tau)$, called the *Bernoulli shift* with distribution p and denoted by $B(p_0, \dots, p_{k-1})$.

II.4. Induced operators:

Very important examples of FDSs arise from TDSs and MDSs as follows:

(1) Let $(X; \varphi)$ be a TDS and let $C(X)$ be the Banach space of all (real- or complex-valued) continuous functions on X (see B.18). Define the "induced operator"

$$T_\varphi : f \mapsto f \circ \varphi \quad \text{for } f \in C(X).$$

It is easy to see that T_φ is an isometric linear operator on $C(X)$, and hence $(C(X); T_\varphi)$ is an FDS. Moreover, we observe that T_φ is a lattice isomorphism (see C.5) and thus a positive operator on the Banach lattice $C(X)$ (see C.1 and C.2). On the other hand, if we consider the complex space $C(X)$ as a C^*-algebra (see C.6 and C.7) it is clear that T_φ is a $*$-algebra isomorphism (see C.8).

(2) Let $(X, \Sigma, \mu; \varphi)$ be an MDS and consider the function spaces $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$ (see B.20). Define

$$T_\varphi : f \mapsto f \circ \varphi \quad \text{for } f \in L^p(X, \Sigma, \mu),$$

or more precisely: $T_\varphi \check{f} := \widetilde{f \circ \varphi}$ where $\check{f}$ denotes the equivalence class in $L^p(X, \Sigma, \mu)$ corresponding to the function f. Again, the "induced operator" T_φ is an isometric (resp. unitary) linear operator on $L^p(X, \Sigma, \mu)$ (resp. on $L^2(\mu)$) since φ is measure-preserving, and hence $(L^p(X, \Sigma, \mu); T_\varphi)$ is an FDS. As above, T_φ is a lattice isomorphism if we consider $L^p(X, \Sigma, \mu)$ as a Banach lattice (see C.1 and C.2). Finally, the space $L^\infty(X, \Sigma, \mu)$ is a commutative C^*-algebra and the induced operator T_φ on $L^\infty(X, \Sigma, \mu)$ is a $*$-algebra isomorphism.

Remark: Via the representation theorem of Gelfand-Neumark the case $(L^\infty(\mu); T_\varphi)$ in (2) may be reduced to the situation of (1) above (see **??**). Therefore we are able to switch from measure-theoretical to functional-analytic or to topological dynamical systems. This flexibility is important in order to tackle a given problem with the most adequate methods.

II.5. Stochastic matrices:

An FDS that is not induced by a TDS or an MDS can be found easily: Take $(E; T)$, where E is $\mathbb{R}^k = C(\{0, \dots, k-1\})$ and T is a $k \times k$-matrix. We single out a particular case or special interest in probability theory: Let T be *stochastic*, i.e. $T = (a_{ij})$ such that $0 \leqslant a_{ij}$ and $\sum_{j=0}^{k-1} a_{ij} = 1$ for $i = 0, 1, \dots, k-1$. Then $(E; T)$ is an FDS and $T\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1, ..., 1)$. The matrix T has the following interpretation in probability theory. We consider $X = \{0, 1, \dots, k-1\}$ as the "state space" of a certain system, and T as a description of time evolution of the states in the following senses a_{ij} denotes the probability that the system moves from state i to state j in one time step and is called the *"transition probability"* from i to j. Thus T (resp. $(E; T)$) can be regarded as a "stochastic" version of a dynamical system. Indeed, if every row and every column of T contains a 1 (and therefore only zeros in the other places), then the system is "deterministic" in the sense that T is induced by a mapping (permutation) $\varphi : X \to X$ (resp. $(E;T)$ is induced by a TDS (X,φ)).

II.6. Markov shifts:
Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be a stochastic matrix (a_{ij}) as in (II.5). Let $\mu = (p_0, \ldots, p_{k-1})^\top$ be an invariant probability vector, i.e.

$$p_i \geqslant 0, \quad \sum_{i=0}^{k-1} p_i = 1$$

and μ is invariant under the adjoint of T, i.e. $\sum_{i=0}^{k-1} a_{ij} p_i = p_j$ for all j (it is well known and also follows from (IV.5) and (IV.4).e that there are such non-trivial invariant vectors). We call μ the *probability distribution* at time 0, and the probabilistic interpretation of the entries a_{ij} (see II.5) gives us a natural way of defining probabilities on

$$\hat{X} := \{0,1,\ldots,k-1\}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \{0,1,\ldots,k-1\}\}$$

with the product σ-algebra $\hat{\Sigma}$. For $0 \leqslant l \leqslant k-1$, $\operatorname{pr}[x_0 = l]$ denotes the probability that $x \in \hat{X}$ is in the state l at time 0. We define

$$\begin{aligned} \operatorname{pr}[x_0 = l] &:= p_l \\ \operatorname{pr}[x_0 = l,\ x_1 = m] &:= p_l a_{lm} \\ \operatorname{pr}[x_0 = l_0,\ x_1 = l_1, \ldots, x_t = l_t] &:= p_{l_0} a_{l_0 l_1} a_{l_1 l_2} \cdots a_{l_{t-1} l_t}. \end{aligned}$$

Moreover, since μ is invariant,

$$\operatorname{pr}[x_1 = l] = \sum_{i=0}^{k-1} \operatorname{pr}[x_0 = i, x_1 = l] = \sum_{i=0}^{k-1} p_i a_{il} = p_l = \operatorname{pr}[x_0 = l],$$

$$\operatorname{pr}[x_t = l] = p_l = \operatorname{pr}[x_0 = l], \quad \text{and finally}$$

$$(*) \qquad \operatorname{pr}[x_s = l_0, x_{s+1} = l_1, \ldots, x_{s+t} = l_t] = p_{l_0} a_{l_0 l_1} a_{l_1 l_2} \cdots a_{l_{t-1} l_t} =$$

$$\operatorname{pr}[x_0 = l_0, x_1 = l_1, \ldots, x_t = l_t] \quad \text{for any choice of } s \in \mathbb{Z},\ t \in \mathbb{N}_0 \text{ and } l_0, \ldots, l_t \in \{0, \ldots, k-1\}$$

The equation $(*)$ gives a probability measure on each algebra $\mathcal{F}_m := \{A \in \hat{\Sigma} : A = \bigcap_{i=-m}^{m} [x_i \in A_i], A_i \subseteq X\}$. By (A.17) this determines exactly one probability measure μ on the product σ-algebra $\hat{\Sigma}$ on $\hat{X}$. This measure μ is obviously invariant under the shift

$$\tau : (x_n) \mapsto (x_{n+1})$$

on $\hat{X}$. Therefore $(\hat{X}, \hat{\Sigma}, \hat{\mu}; \tau)$ is an MDS, called the *Markov shift* with *invariant distribution* μ and *transition matrix* T.

Note that the examples (II.5) and (II.6), although they describe the same stochastic process, are quite different, because the operator T of (II.5) is not induced by a transformation of the state space $\{0, 1, \ldots, k-1\}$, whereas in (II.6) the shift τ is defined on the state space $\{0, 1, \ldots, k-1\}^{\mathbb{Z}}$. We have refined (i.e. enlarged) the state space of (II.5) to make the model "deterministic".

An analogous construction can be carried out in the infinite-dimensional case for so-called Markov-operators (see App. U and X), or for transition probabilities (see Bauer).

This construction is well-known in the theory of Markov processes; its functional-analytic counterpart, the so-called *dilation*, will be presented in App. U.

Exercise: The Bernoulli shift $B(p_0, \dots, p_{k-1})$ is a Markov shift. What is its invariant distribution and its transition matrix?

II.D Discussion

II.D.1. Non-bijective dynamical systems:
It is clear, that the Definitions (II.1.i,ii) make sense not only for bijective but also for arbitrary measure-preserving, resp. continuous transformations, but we prefer to sacrifice this greater generality for the sake of simplicity. Such non-bijective transformations also induce FDSs by a procedure similar to that in (II.4). Examples are the mappings

$$\varphi : [0,1] \to [0,1] \quad \text{defined by}$$

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ 2 - 2t & \text{for } \frac{1}{2} < t \leqslant 1 \end{cases}$$

or

$$\varphi(t) := 4t(1-t).$$

II.D.2. Banach algebras vs. Banach lattices:
The function spaces used in ergodic theory, i.e. $C(X)$ and $L^p(X, \Sigma, \mu)$, are Banach lattices and the induced operators T_φ are lattice isomorphisms (see II.4 and App.C). Therefore, the vector lattice structure seems to be adequate for a simultaneous treatment of topological and measure-theoretical dynamical systems. If you prefer Banach algebras and algebra isomorphisms, you have to consider the operators T_φ on the spaces $C(X)$ and $L^\infty(X, \Sigma, \mu)$.

II.D.3. Real vs. complex Banach spaces:
Since order structure and positivity makes sense only for real Banach spaces, one could be inclined to study only spaces of *real* valued functions. But methods from spectral theory play a central role in ergodic theory and require *complex* Banach spaces. However, no real trouble is caused, since the complex Banach spaces $C(X)$ and $L^p(X, \Sigma, \mu)$ decompose canonically into real and imaginary parts, and we restrict our attention to the real part whenever we use the order relation. Moreover, the induced operator T_φ (like any positive linear operator) is uniquely determined by its restriction to this real part.

II.D.4. Null sets in (X, Σ, μ):
In the measure-theoretical case some technical problems may be caused by the sets $A \in \Sigma$ with $\mu(A) = 0$. But in ergodic theory, it is customary (and reasonable, as can be understood from the physicist's answer in Lecture I: A is a set of "states" having probability 0) to identify measurable sets which differ only by such a null set. From now on, this will be done without explicit statement. For example, we will say that a measurable function f is constant if

$$f(x) = c$$

for all $x \in X\backslash A$, $\mu(A) = 0$.
The reader familiar with the "function" spaces $L^p(X, \Sigma, \mu)$ realizes that we identify the function with its equivalence class in $L^p(\mu)$, but still keep the terminology of functions. These subtleties should not disturb the beginner since no serious mistakes can be made (see A.7 and B.20).

II.D.5. Which FDSs are TDSs?
We have seen in II.4 that to every TDS $(X;\varphi)$ canonically corresponds the FDS $(C(X), T_\varphi)$. Since this correspondence occurs frequently in our operator-theoretical approach to ergodic theory, it is important to know which FDSs arise in this way. More precisely: Which operators

$$T : C(X) \to C(X)$$

are induced by a homeomorphism

$$\varphi : X \to X$$

in the sense that $T = T_\varphi$? A complete answer is given as follows.

Theorem: Consider the real Banach space $C(X)$ and $T \in \mathscr{L}(C(X))$. Then the following assertions are equivalent:

(i) T is a lattice isomorphism satisfying $T\mathbf{1} = \mathbf{1}$.
(ii) T is an algebra isomorphism.
(iii) $T = T_\varphi$ for a (unique) homeomorphism φ on X.

Proof. Clearly, (iii) implies (i) and (ii).

(ii) $\Rightarrow$ (iii): Let $D := \{\delta_x : x \in X\}$ be the weak* compact set of all Dirac measures on X. This coincides with the set of all normalized multiplicative linear forms on $C(X)$, and from (C.9) it follows that X is homeomorphic to D. Since T is an algebra isomorphism its adjoint T' maps D on D. The restriction of T' to D defines a homeomorphism φ on X having the desired properties.

(i) $\Rightarrow$ (iii): The proof requires some familarity with Banach lattices. We refer to Schaefer 1974, III.9.1 for the details as well as for the "complex" case of the theorem. ∎

II.D.6. Which FDSs are MDSs?
Due to the existence of null sets (and null functions) the analogous problem in the measure-theoretical context is more difficult: Which operators

$$T : L^p(X, \Sigma, \mu) \to L^p(X, \Sigma, \mu)$$

are induced by a bi-measure-preserving transformation

$$\varphi : X \to X$$

in the sense that $T = T_\varphi$? Essentially, it turns out that the appropriate operators are again the Banach lattice isomorphisms, but we will return to this problem in Lecture VI.

II.D.7. Discrete vs. continuous time:
Applying φ (or T) in a dynamical system may be interpreted as movement from the state x at time t to the state $\varphi(x)$ at time $t + \Delta t$. Therefore, repeated application of φ means advancing in time with a discrete time scale in steps of Δt. Intuitively it is more realistic to consider a continuous time scale, and in our mathematical model the transformation φ and the group homomorphism

$$n \mapsto \varphi^n$$

defined on $\mathbb{Z}$ should be replaced by a continuous group of transformations, i.e. a group homomorphism

$$t \mapsto \varphi_t$$

from $\mathbb{R}$ into an appropriate set of transformations on X. Observe that the "composition rule"

$$\varphi^{n+m} = \varphi^n \circ \varphi^m, \quad n, m \in \mathbb{Z},$$

in the discrete model is replaced by

$$\varphi_{t+s} = \varphi_t \circ \varphi_s, \quad t, s \in \mathbb{R}.$$

Adding some continuity or measurability assumptions one obtains "continuous dynamical systems" (e.g. Rohlin [1966], Chapt. II.). We prefer the simpler discrete model, since we are mainly interested in the asymptotic behavior of the system as t tends to infinity.

II.D.8. From a differential equation to a dynamical system:
In (II.D.7) we briefly discussed the problem "discrete vs.continuous time". Clearly, a "continuous dynamical system" $(X; (\varphi_t)_{t\in\mathbb{R}})$ gives rise to many "discrete dynamical systems" $(X; \varphi)$ by setting $\varphi := \varphi_t$ for any $t \in \mathbb{R}$. We present here a short introduction into the so-called "classical dynamical systems" which arise from differential equations and yield continuous dynamical systems, also called "flows".

Let $X \subseteq \mathbb{R}^n$ be a compact smooth manifold and $f(x)$ a C^1-vector field on X. We consider the autonomous ordinary differential equation

$$(*) \qquad \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

(or in coordinates: $\dot{x}_i = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$). It is known that for every $x \in X$ the equation $(*)$ has a unique solution $\varphi_t(x)$ that satisfies $\varphi_0(x) = x$. The uniqueness of the solution implies the group property $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \in \mathbb{R}$, and, in addition, the mapping

$$\begin{aligned} \Phi : \times\mathbb{R} &\to X \\ (x, t) &\mapsto \varphi_t(x) \end{aligned}$$

is continuous (see Nemyckii-Stepanov [1960]). Therefore, $(X; (\varphi_t)_{t\in\mathbb{R}})$ is a continuous topological dynamical system.

II.D.9 Examples:
(i) Let $\Gamma^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and let

$$\begin{aligned} \dot{x} &= 1 \\ \dot{y} &= \alpha \end{aligned}$$

with $\alpha \neq 0$. The flow (φ_t) on Γ^2 is given by

$$\varphi_t\big(\binom{x}{y}\big) = \begin{pmatrix} (x+t) \mod 1 \\ (y+\alpha t) \mod 1 \end{pmatrix}.$$

(ii) Take the space $X = \Gamma^2$ as in (i) and define

$$\dot{x} = F\big(\binom{x}{y}\big)$$
$$\dot{y} = \alpha \cdot F\big(\binom{x}{y}\big)$$

where F is C^1-function which is 1-periodic in each variable. Assume that F is strictly positive on X. The solution curves of this motion agree with those of (i), but the "speed" is changed.

For applications the above definition of a "continuous topological dynamical system" has three disadvantages: first, the manifold X (the "state" space) is not always compact, second, if X is not compact, in general not every-solution of $(*)$ can be continued for all times t (e.g. the scalar equation $\dot{x} = x^2$), and finally, it is often necessary to consider non-autonomous differential equations, i.e. the C^1-vector field f is defined on $X \times \mathbb{R}$ where X is a manifold. All of these difficulties can be overcome by generalizing the above definition (see Sell [1971].

Next, we want to consider "classical measure-theoretical dynamical systems". The problem of finding a φ_t-invariant measure, defined by a continuous density, is solved by the Liouville theorem (see Nemyckii-Stepanov [1960]). We only present a special case.

Many equations of classical mechanics can be written as a Hamiltonian system of differential equations. Let $q = (q_1, \ldots, q_n)$ (coordinates) and $p = (p_1, \ldots, p_n)$ (moments) be a coordinate system in $\mathbb{R}^{2n}$ and $H(p,q)$ a C^2-function which does not depend on time explicitly. The equations

$$(**) \qquad \begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} \end{aligned}$$

define a flow on $\mathbb{R}^{2n}$ called the "Hamiltonian flow". The divergence of the vector field $(**)$ vanishes:

$$\frac{\partial}{\partial q}\Big(\frac{\partial H}{\partial p}\Big) + \frac{\partial}{\partial p}\Big(\frac{\partial H}{\partial q}\Big) = 0.$$

Therefore, the measure $\mathrm{d}q_1 \ldots \mathrm{d}q_n\, \mathrm{d}p_1 \ldots \mathrm{d}p_n$is invariant under the induced flow. But the considered state space is not compact and the invariant measure is not finite.

To avoid this difficulty we observe that

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} + \frac{\partial H}{\partial q}\Big(-\frac{\partial H}{\partial q}\Big) = 0$$

i.e. H is a first integral of $(**)$ (conservation of energy!). This means that $X_E := \{(p,q) \in \mathbb{R}^{2n} : H(p,q) = E\}$ for every $E \in \mathbb{R}$ is invariant under the flow. X_E turns out to be a compact smooth manifold for typical values of the constant E, and we obtain on it an "induced" measure by a method similar to the construction of the 1-dimensional Lebesgue measure from the 2-dimensional Lebesgue measure. This induced measure is (φ_t)-invariant and finite, and we obtain "continuous measure-theoretical dynamical systems".

Example (linear harmonic oscillator): Let $X = \mathbb{R}^2$ and let $\binom{p}{q}$ be the canonical coordinates on X. For simplicity, we suppose that the constants of the oscillator are all 1. The Hamiltonian function is the sum of the kinetic and the potential energy and therefore

$$H(p,q) = H_{\text{kin}}(p) + H_{\text{pot}}(q) = \frac{1}{2}p^2 + \frac{1}{2}q^2$$

The system $(**)$ becomes

$$\begin{aligned} \dot{q} &= \quad p \\ \dot{p} &= -q \end{aligned}$$

and the solution with initial value $\binom{p}{q}$ is

$$\varphi_t\left(\binom{p}{q}\right) = \begin{pmatrix} \sqrt{p^2+q^2}\,\sin(t+\beta) \\ \sqrt{p^2+q^2}\,\cos(t+\beta) \end{pmatrix},$$

where $\beta \in [0, 2\pi)$ is defined by $\sqrt{p^2+q^2}\cdot\sin\beta = q$ and $\sqrt{p^2+q^2}\cdot\cos\beta = p$. Now, let us consider the surface $H(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2 =: E =$constant.

Obviously, E must be positive. For $E = 0$ we have the (invariant) trivial manifold $\{\binom{0}{0}\}$. For $E > 0$ the $(\varphi_t)_{t\in\mathbb{R}}$-invariant manifold

$$X_E := \left\{\binom{p}{q} \in \mathbb{R}^2 : H(p,q) = E\right\}$$

is the circle about 0 with radius $\sqrt{2E}$, and therefore compact. The "induced" invariant measure on X_E is the 1-dimensional Lebesgue measure, and the induced flow agrees with a flow of rotations on this circle.

II.D.10. Dilating an FDS to an MDS:

We have indicated in (II.D.6) that rather few FDSs on Banach spaces $L^1(\mu)$ are induced by MDSs. But in (II.6) we presented an ingenious way of reducing the study of certain FDSs to the study of MDSs. These constructions are solutions of the following problem:

Let T be a bounded linear operator on $E = L^1(X, \Sigma, \mu)$, $\mu(X) = 1$. Can we find an MDS $(\hat{X}, \hat{\Sigma}, \hat{\mu}; \varphi)$ and operators J and Q, such that the diagram

$$\begin{array}{ccc} L^1(X,\Sigma,\mu) & \xrightarrow{\;T^n\;} & L^1(X,\Sigma,\mu) \\ {\scriptstyle J}\downarrow & & \uparrow{\scriptstyle Q} \\ L^1(\hat{X},\hat{\Sigma},\hat{\mu}) & \xrightarrow[\hat{T}_\varphi^n]{} & L^1(\hat{X},\hat{\Sigma},\hat{\mu}) \end{array}$$

commutes for all $n = 0, 1, 2, \ldots$?

If we want the MDS $(\hat{X}, \hat{\Sigma}, \hat{\mu}; \varphi)$ to reflect somehow the "ergodic" behaviour of the FDS $(L^1(X, \Sigma, \mu); T)$, it is clear that the operators J and Q must preserve the order structure of the L^1-spaces (see II.4). Therefore, we call $(L^1(\hat{X}, \hat{\Sigma}, \hat{\mu}); \hat{T}_\varphi)$, resp. $(\hat{X}, \hat{\Sigma}, \hat{\mu}; \varphi)$, a *lattice dilation* of $(L^1(X, \Sigma, \mu); T)$ if - in the diagram above - J is an isometric lattice homomorphism (with $J\mathbf{1} = \hat{\mathbf{1}}$), and Q is a positive contraction. From these requirements it follows that T has to be positive with $T\mathbf{1} = \mathbf{1}$ and $T'\mathbf{1} = \mathbf{1}$. In App. U we show that these conditions are even sufficient.