Appendix S. Invariant Measures

If $(X;\varphi)$ is a TDS it is important to know whether there exists a probability measure ν on X which is invariant under φ. Such an invariant measure allows the application of the measure-theoretical results in the topological context. It is even more important to obtain a φ-invariant measure on X which is equivalent to a particular probability measure (e.g. to the Lebesgue measure). The following two results show that the answer to the first question is always positive while the second property is equivalent to the mean ergodicity of some induced linear operator.

S.1 Theorem (Krylov-Bogoliubov, 1937):
Let X be compact and $\varphi : X \to X$ continuous. There exists a probability measure $\nu \in C(X)'$ which is φ-invariant.

Proof. Consider the induced operator $T := T_\varphi$ on $C(X)$. Its adjoint T' leaves invariant the weak*-compact set $\mathscr{P}$ of all probability measures in $M(X)$. If $\nu_0 \in \mathscr{P}$, then the sequence $\{T_n'\nu_0 : n \in \mathbb{N}\}$ has a weak∗-accumulation point ν. It is easy to see (use IV.3.0) that $T'\nu = \nu$, i.e. ν is φ-invariant. ■

As a consequence we observe that every TDS $(X;\varphi)$ may be converted into an MDS $(X, \mathcal{B}, \mu; \varphi)$ where $\mathcal{B}$ is the Borel algebra and μ some φ-invariant probability measure. Moreover, the set $\mathscr{P}_\varphi$ of all φ-invariant measures in $\mathscr{P}$ is a convex $\sigma(C(X)', C(X))$-compact subset of $C(X)'$. Therefore, the Krein-Milman theorem yields many extreme points of $\mathscr{P}_\varphi$ called "ergodic measures". The reason for that nomenclature lies in the following characterization.

S.2 Corollary:
Let $(X;\varphi)$ be a TDS. μ is an extreme point of $\mathscr{P}_\varphi$ if and only if $(X, \mathcal{B}, \mu; \varphi)$ is an ergodic MDS.

Proof. If $(X, \mathcal{B}, \mu; \varphi)$ is not ergodic there exists $A \in \mathcal{B}$, $0 < \mu(A) < 1$, such that $\varphi(A) = A$ and $\varphi(X\backslash A) = X\backslash A$. Define two different measures

$$\mu_1(B) := \frac{\mu(B \cap A)}{\mu(A)}$$
$$\mu_2(B) := \frac{\mu(B \cap (X\backslash A))}{\mu(X\backslash A)} \quad \text{for } B \in \mathcal{B}.$$

Clearly, $\mu = \mu(A) \cdot \mu_1 + (1 - \mu(A)) \cdot \mu_2$, and μ not an extreme point of $\mathscr{P}_\varphi$.

On the other hand, assume $(X, \mathcal{B}, \mu; \varphi)$ to be ergodic. If $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ for $\mu_1, \mu_2 \in \mathscr{P}_\varphi$, then $\mu_1 \leqslant 2\mu$ and hence $\mu_1 \in L^1(\mu)' = L^\infty(\mu)$. But the fixed space of T_φ' in $L^\infty(\mu)$ contains μ and $\mu_1$and is one-dimensional by (IV.6), (IV.4.e) and (III.4). Therefore we conclude $\mu = \mu_1$, i.e. μ must be an extreme point of $\mathscr{P}_\varphi$. ■

The question, whether there exist φ-invariant probability measures equivalent to some distinguished measure, is more difficult and will be converted into a "mean ergodic" problem.

S.3 Theorem:
Let μ be a strictly positive probability measure on some compact space X and let $\varphi : X \to X$ be Borel measurable and non-singular with respect to μ (i.e. $\mu(A) = 0$ implies $\mu(\varphi^{-1}(A)) = 0$ for $A \in \mathcal{B}$). The following conditions are equivalent:

(a) There exists a φ-invariant probability measure ν on X which is equivalent to μ.
(b) For the induced operator $T := T_\varphi$ on $L^\infty(X, \mathcal{B}, \mu)$ the Cesàro means T converge in the $\sigma(L^\infty, L^1)$-operator topology to some strictly positive projection $P \in \mathscr{L}(L^\infty(\mu))$, i.e. $Pf > 0$ for $0 < f \in L^\infty$.
(c) The pre-adjoint T' of $T = T_\varphi$ is mean ergodic on $L^1(\mu)$ and $T'u = u$ for some strictly positive $u \in L^1(\mu)$.

Proof. The assumptions on φ imply that $T = T_\varphi$ is a well-defined positive contraction on $L^\infty(\mu)$ having a pre-adjoint T' on $L^1(m)$ (see Schaefer [1974], III.9, Example 1).

(a) $\Rightarrow$ (c): By the Radon-Nikodym theorem the φ-invariant probability measure ν equivalent to μ corresponds to a normalized strictly positive T-invariant function $u \in L^1(\mu)$. But for such functions the order interval

$$[-u, u] := \{f \in L^1(\mu) : -u \leqslant f \leqslant u\}$$

is weakly compact and total in $L^1(\mu)$. Therefore $Tu = u$ implies the mean ergodicity of T as in (IV.6).

(c) implies (b) by a simple argument using duality theory.

(b) $\Rightarrow$ (a): The projection $P : L^\infty(\mu) \to L^\infty(\mu)$ satisfies $PT = TP = P$ and maps $L^\infty(\mu)$ onto the T-fixed space. Consider

$$\nu_0 := \mu \circ P$$

which is a strictly positive φ-invariant linear form on $L^\infty(\mu)$. Since the dual of $L^\infty(\mu)$ decomposes into the band $L^1(\mu)$ and its orthogonal band we may take ν as the band component of ν_0 in $L^1(\mu)$.

By Ando [1968], Lemma 1, ν is still strictly positive and hence defines a measure equivalent to μ. Moreover, $T'\nu$ is contained in $L^1(\mu)$ and dominated by ν_0, hence $T'\nu \leqslant \nu$. From $T\mathbf{1} = \mathbf{1}$ we conclude $T'\nu = \nu$ and that ν is φ invariant. Normalization of ν yields the desired probability measure. ∎

These abstract results are not only elegant and satisfying from a theoretical standpoint, they can also help to solve rather concrete problems:

Let $\varphi : [0,1] \to [0,1]$ be a transformation which is *piecewise* C^2, i.e. there is a finite partition of $[0,1]$ in intervals A_i such that φ can be extended continuously from the interior $\mathring{A}_i$ to the closure $\overline{A_i}$ and the resulting function φ_i is twice continuously differentiable on $\overline{A_i}$. Moreover we assume that the derivatives $\dot{\varphi}_i$ do not vanish on $\mathring{A}_i$, φ_i is increasing or decreasing.

In this case, φ is measurable and non-singular with respect to the Lebesgue measure m, and

$$Tf := f \circ \varphi$$

defines a positive contraction on $L^\infty([0,1], \mathcal{B}, m)$ satisfying $T\mathbf{1} = \mathbf{1}$ and having a pre-adjoint T' on $L^1(m)$.
As a consequence of this theorem, one concludes that φ possesses an invariant probability measure which is absolutely continuous with respect to m iff $\dim F(T') \geqslant 1$. In particular, this follows if T' is mean ergodic.

To find out under which conditions on φ this holds, we observe that the pre-adjoint T' can be written as

$$T'f(x) = \sum_i f \circ \varphi_i^{-1}(x)\sigma_i(x)\mathbf{1}_{B_i}(x),$$

where $B_i = \varphi_i(\overline{A_i})$ and σ_i is the absolute value of the derivative of φ_i^{-1}.

In fact: For every $x \in (0,1)$,

$$\int_0^x T'f \, \mathrm{d}m = \int_0^1 f \cdot \mathbf{1}_{(0,x)} \circ \varphi \, \mathrm{d}m = \int_{\varphi^{-1}(0,x)} f \, \mathrm{d}m.$$

Thus $T'f$ is the derivative $\dot{g}$ of the function $g(x) = \int_{\varphi^{-1}(0,x)} f \, \mathrm{d}m$.

If φ is piecewise C^2, we can calculate this derivative and obtain the above formula. Recall that the *variation* $v(f)$ of a function $f : [a,b] \to \mathbb{R}$ is defined as

$$v(f) := \sup_{n\in\mathbb{N}} \Big\{ \sum_{j=1}^n |f(t_j) - f(t_{j-1})| : a = t_0 < t_1 \cdots < t_n = b \Big\}.$$

With this concept and using some elementary analysis, one proves that

$$(*) \qquad v(f \cdot g) \leqslant v(f)\|g\|_\infty + \int_a^b |f \cdot \dot{g}| \, \mathrm{d}m$$

if f is piecewise continuous and g continuously differentiable.

After these preparations we present the main result.

S.4 Proposition:
Let $\varphi : [0,1] \to [0,1]$ be piecewise C^2 such that

$$s := \inf\big\{|\dot{\varphi}(t)| : t \in (0,1) \text{ and } \varphi \text{ differentiable at } t\big\} > 1.$$

Then there exists a φ-invariant probability measure on $[0,1]$ which is absolutely continuous with respect to the Lebesgue measure m.

Proof. By (S.3) we have to show that the pre-adjoint T'_φ of T_φ is mean ergodic on $L^1(m)$. The first part of the proof is of a technical nature. Choose $n \in \mathbb{N}$ such that $s^n > 2$ and consider the map

$$\Phi := \varphi^n$$

which again is piecewise C^2. Clearly,

$$\inf\{|\dot{\Phi}(t)| : t \in (0,1) \text{ and } \Phi \text{ differentiable at } t\} \geqslant s^n > 2.$$

Now we estimate the variation $v(T'_\varphi f)$ for any piecewise continuous function $f : [0,1] \to \mathbb{R}$. To this purpose we need some constants determined by the function Φ. Take the partition of $[0,1]$ into intervals A_i corresponding to φ and write

$$T'_\Phi f(x) = \sum_{i=1}^m f \circ \Phi_i^{-1}(x)\sigma_i(x)\mathbf{1}_{B_i}(x)$$

where $B_i = \Phi_i(\overline{A_i})$ and $\sigma_i(x) = |(\dot{\Phi}_i^{-1})(x)|$.

1. For σ_i we have $\sigma_i(x) \leqslant s^{-n} \leqslant \frac{1}{2}$ for every $x \in B_i$.
2. Put $k := \max\big\{|\dot{\sigma}_i(x)| : x \in \overline{B_i};\ i = 1,\dots,m\big\} \cdot \max\big\{|\dot{\Phi}_i(x)| : x \in \overline{A_i};\ i = 1,\dots,m\big\}$.

3. For the interval $\overline{A_i} = [a_{i-1}, a_i]$ we estimate

$$\begin{aligned}|f(a_{i-1})| + |f(a_i)| &\leqslant 2\inf\{|f(x)| : x \in A_i\} + v(f|_{A_i}) \\ &\leqslant \frac{2}{m(A_i)} \int_{a_i} |f|\, \mathrm{d}m + v(f|_{A_i}) \\ &\leqslant 2h \int_{A_i} |f|\, \mathrm{d}m + +v(f|_{A_i})\end{aligned}$$

for $h := \max\{\frac{1}{m(A_i)} : i = 1, \ldots, m\}$.

Now, we can calculate:

$$\begin{aligned}v(T'_\Phi f) &\leqslant \sum_{i=1}^{m} v\big(f \circ \Phi_i^{-1}(x)\sigma_i(x) \cdot \mathbf{1}_{B_i}(x)\big) \\ &\leqslant \sum_{i=1}^{m} \Big(\|\sigma_i\|_\infty \cdot v\big(f \circ \Phi_i^{-1}(x) \cdot \mathbf{1}_{B_i}(x)\big) + \int_{B_i} \big|f \circ \Phi_i^{-1} \cdot \dot{\sigma}_i\big|\, \mathrm{d}m\Big)\end{aligned}$$

(by inequality $(*)$ above)

$$\leqslant \sum_{i=1}^{m} \Big(s^{-n}(|f(a_{i-1})| + |f(a_i)| + v(f|_{A_i})) + k \int_{B_i} |f \circ \Phi_i^{-1}| \cdot \sigma_i\, \mathrm{d}m\Big)$$

(since $\max\{|\dot{\Phi}_i(x)| : x \in \overline{A_i}; i = 1, \ldots, m\} = \min\{\sigma_i(x) : x \in B_i; i = 1, \ldots, m\}$)

$$\begin{aligned}&\leqslant \sum_{i=1}^{m} \Big(s^{-n}(2h \int_{a_i} |f|\, \mathrm{d}m + 2v(f|_{A_i})) + k \int_{A_i} |f|\, \mathrm{d}m\Big) \\ &\leqslant (h + k)\|f\|_1 + 2s^{-n} v(f).\end{aligned}$$

Observing that $v(\mathbf{1}) = 0$ and ${T'_\Phi}^r \mathbf{1}$ is again piecewise continuous, we obtain by induction

$$v({T'_\Phi}^r \mathbf{1}) \leqslant (h + k) \sum_{i=0}^{r-1} (2s^{-n})^i \leqslant \frac{h + k}{1 - 2s^{-n}} \quad \text{for every } r \in \mathbb{N},$$

and therefore

$$\|{T'_\Phi}^r \mathbf{1}\|_\infty \leqslant \|{T'_\Phi}^r \mathbf{1}\|_1 + v({T'_\Phi}^r \mathbf{1}) \leqslant 1 + \frac{h + k}{1 - 2s^{-n}},$$

i.e. ${T'_\Phi}^r \mathbf{1} \leqslant M \cdot \mathbf{1}$ for $r \in \mathbb{N}$ and some $M > 0$. For the final conclusion the abstract mean ergodic theorem (IV.6) implies that T'_Φ is mean ergodic. Since $T'_\Phi = {T'_\varphi}^n$, the same is true for T'_φ by (IV.D.2). ∎

In conclusion, we present some examples showing the range of the above proposition.

S.5 Examples:

1. The transformation

$$\varphi(t) := \begin{cases} 2t & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ 2 - 2t & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

satisfies the assumptions of our proposition and has a φ-invariant measure. In fact, m itself is invariant.

2. For

$$\varphi(t) := \begin{cases} \frac{t}{1-t} & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ 2t-1 & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

The assumption $|\dot{\varphi}(t)| > 1$ is violated at $t = 0$. In fact, there is no φ-invariant and with respect to m absolutely continuous measure on $[0,1]$, since ${T'_\varphi}^n f$ converges to 0 in measure for $f \in L^1(m)$ (see Lasota-Yorke [1973]).

3. For $\varphi(t) := 4t \cdot (1-t)$ is strongly violated, nevertheless there is a φ-invariant measure: Indeed, the equation $\int_{[0,x]} f \,\mathrm{d}m = \int_{\varphi^{-1}[0,x]} f \,\mathrm{d}m$ together with the plausible assumption that $f(t) = f(1-t)$ leads to

$$F(x) := \int_0^x f(t)\,\mathrm{d}t = 2 \cdot \int_0^{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}} f(t)\,\mathrm{d}t = 2 \cdot F(\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-x}).$$

By substituting $x = \sin^2 \xi$ we obtain

$$F(\sin^2 \xi) = 2F(\tfrac{1}{2} - \tfrac{1}{2}\cos\xi) = 2F(\sin^2 \tfrac{\xi}{2})$$

which shows that $F(x) = \arcsin\sqrt{x}$ is a solution. Thus the function

$$f(x) = \frac{1}{2\sqrt{x(1-x)}}$$

yields a φ-invariant measure $f \cdot m$ on $[0,1]$

4. Finally, $\varphi(t) := 2(t - 2^{-i}$ for $2^{-i} < t \leqslant 2^{1-i}$, $i \in \mathbb{N}$, has $\dot{\varphi}_i(t) = 2$, but infinitely many discontinuities. Again there exists no φ-invariant measure since ${T'_\varphi}^n f$ converges to zero in measure for $f \in L^1(m)$.

References: Ando [1968], Bowen [1979], Brunel [1970], Hajian-Ito [1967], Lasota [1980], Lasota-Yorke [1973], Neveu [1967], Oxtoby [1952], Pianigiani [1979], Takahashi [1971].