Appendix E. Some Analytic Lemmas

Here, we prove some analytic lemmas which we use in the present lectures but don't prove there in order not to interrupt the main line of the arguments. First, we recall two definitions.

E.1 Definition:

1. A sequence $(x_n)_{n\in\mathbb{N}}$ of real (or complex) numbers is called Cesàro-summable if $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} x_i$ exists.
2. Let $(n_i)_{i\in\mathbb{N}}$ be a subsequence of $\mathbb{N}_0$. Then $(n_i)_{i\in\mathbb{N}}$ has *density* $s \in [0,1]$, denoted by $d((n_i)_{i\in\mathbb{N}}) = s$, if

$$\lim_{k\to\infty}\frac{1}{k}\big|\{n_i : i \in \mathbb{N}\} \cap \{0,1,\ldots,k-1\}\big| = s$$

where $|\cdot|$ denotes the cardinality.

E.2 Lemma:
For $(x_n)_{n\in\mathbb{N}_0}$ the following conditions are equivalent:

(i) $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|x_i| = 0.$

(ii) There exists a subsequence N of $\mathbb{N}_0$ with $d(N) = 1$ such that $\lim_{\substack{n\in N\\ n\to\infty}} x_n = 0.$

Proof. We define $N_k := \{0,1,\ldots,k-1\}$.
(i) $\Rightarrow$ (ii): Let $J_k := \{n \in \mathbb{N}_0 : |x_n| \geqslant \frac{1}{k}\}$, $k > 0$, and observe that $J_1 \subseteq J_2 \subseteq \cdots$. Since $\frac{1}{n}\sum_{i=0}^{n-1}|x_i| \geqslant \frac{1}{n}\cdot\frac{1}{k}|J_k \cap N_n|$, each J_k has density 0. Therefore, we can choose integers $0 = n_0 < n_1 < n_2 < \cdots$ such that

$$\frac{1}{n}|J_{k+1} \cap N_n| < \frac{1}{k+1} \qquad \text{for } n \geqslant n_k.$$

Define $J := \bigcup_{k\in\mathbb{N}}(J_{k+1} \cap (N_{n_{k+1}}\backslash N_{n_k}))$ and show $d(J) = 0$.
Let $n_k \leqslant n < n_{k+1}$. Then, we obtain

$$J \cap N_n = (J \cap N_{n_k}) \cup (J \cap (N_n\backslash N_{n_k})) \subseteq (J_k \cap N_{n_k}) \cup (J_{k+1} \cap N_n),$$

and conclude that

$$\frac{1}{n}|J \cap N_n| \leqslant \frac{1}{k} + \frac{1}{k+1}.$$

If n tends to infinity, the same is true for k, and hence, J has density 0. Obviously, the sequence $N := \mathbb{N}\backslash J$ has the desired properties.

(ii) $\Rightarrow$ (i): Let $\varepsilon > 0$ and $c := \sup\{|x_n| : n \in \mathbb{N}_0\}$. Because of (ii) and $d(\mathbb{N}\backslash N) = 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $n \geqslant n_\varepsilon$ implies $|x_n| < \varepsilon$ for $n \in N$ and $\frac{1}{n}|(\mathbb{N}\backslash N)\cap N_n| <$

ε. If $n \geqslant n_\varepsilon$ we conclude that

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1}|x_i| &= \frac{1}{n}\sum_{i\in(\mathbb{N}\backslash N)\cap N_n}|x_i| + \frac{1}{n}\sum_{i\in N\cap N_n}|x_i| \\ &\leqslant \frac{c}{n}|(\mathbb{N}\backslash N)\cap N_n| + \varepsilon \\ &\leqslant (c+1)\cdot\varepsilon.\end{aligned}$$

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E.3 Lemma:
Take a sequence $(z_n)_{n\in\mathbb{N}}$ of complex numbers such that

$$\sum_{n=1}^{\infty} n|z_{n+1}-z_n|^2 < \infty.$$

If $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} z_i = 0$, then $\lim_{n\to\infty} z_n = 0$.

Proof. Define $c_n := \sum_{k=n}^{\infty} k|z_{k+1}-z_k|^2$. Then

$$\begin{aligned}\max\{|z_{n+k}-z_n| : 1\leqslant k\leqslant n-2\} \leqslant \sum_{k=n}^{2n-3}|z_{k+1}-z_k| &\leqslant \Big(\sum_{k=n}^{2n-3}|z_{k+1}-z_k|^2(n-2)\Big)^{1/2} \\ &\leqslant c_n\end{aligned}$$

and $$|z_n| = \Big|b_{n-1} - 2b_{2n-2} + \frac{1}{n-1}\sum_{k=1}^{n-2}(z_{n+k}-z_n)\Big| \quad \text{for } b_n := \frac{1}{n}\sum_{i=1}^{n} z_i.$$

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E.4 Lemma:
Let N_i, $i = 1, 2, \ldots$ be a subsequence of $\mathbb{N}_0$ with density $d(N_i) = 1$. Then there exists a subsequence N of $\mathbb{N}_0$ such that $d(N) = 1$ and $N\backslash N_i$ is finite for every $i \in \mathbb{N}$.

Proof. There exists an increasing sequence $(k_i)_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$1 - 2^{-i} \leqslant \frac{1}{k}\big|N_i \cap \{0,\ldots,k-1\}\big| \quad \text{for all } k \geqslant k_i.$$

If we define $N := \bigcap_{i\in\mathbb{N}} N_i \cup \{0,\ldots,k_i-1\}$, then N has the desired properties. ■

E.5 Lemma:
If $(x_n)_{n\in\mathbb{N}}$ is a sequence of psoitive reals satisfying $x_{n+m} \leqslant x_n + x_m$ for all $n, m \in \mathbb{N}$, then $\lim_{n\to\infty}\frac{x_n}{n}$ exists and equals $\inf_{n\in\mathbb{N}}\frac{x_n}{n}$.

Proof. Fix $n > 0$, and for $j > 0$ write $j = kn + m$ where $k \in \mathbb{N}_0$ and $0 \leqslant m < n$. Then

$$\frac{x_j}{j} = \frac{x_{kn+m}}{kn+m} \leqslant \frac{x_{kn}}{kn} + \frac{x_m}{kn} \leqslant \frac{kx_n}{kn} + \frac{x_m}{kn} = \frac{x_n}{n} + \frac{x_m}{kn}.$$

If $j \to \infty$ then $k \to \infty$, too, and we obtain

$$\limsup_{j\to\infty}\frac{x_j}{j} \leqslant \frac{x_n}{n}, \quad \text{and even} \quad \limsup_{j\to\infty}\frac{x_j}{j} \leqslant \inf_{n\in\mathbb{N}}\frac{x_n}{n},$$

On the other $\inf_{n\in\mathbb{N}}\frac{x_n}{n} \leqslant \liminf_{n\to\infty}\frac{x_n}{n}$, and the lemma is proved. ■