Appendix D. Remarks on Compact Commutative Groups

Important examples in ergodic theory are obtained by rotations on compact groups, in particular on the tori Γ^n. In our Lectures VII and VIII we use some facts about compact groups and character theory of locally compact abelian groups. Therefore, we mention the basic definitions and main results and refer to Hewitt-Ross [1979] for more information.

D.1. Topological groups:

A group $(G, \cdot)$ is called a *topological group* if it is a topological space and the mappings

$$(g,h) \mapsto g \cdot h \qquad \text{on } G \times G$$

and

$$g \mapsto g^{-1} \qquad \text{on } G$$

are continuous. A topological group is a *compact group* if G is compact. An isomorphism of topological groups is a group isomorphism which simultaneously is a homeomorphism.

D.2. The Haar measure:

Let G be a compact group. Then there exists a unique (right and left) invariant probability measure m on G, i.e. $= R_g'm = L_g'm$ for all $g \in G$ where R_g denotes the right rotation $R_g f(x) := f(xg)$, $x \in G$, $f \in C(G)$, and L_g the left rotation on $C(G)$.
m is called the *normalized Haar measure* on G.
The existence of Haar measure on compact groups can be proved using mean ergodic theory (e.g. (**??**.1) or Schaefer [1977], III.7.9, Corollary 1). For a more general and elementary proof see Hewitt-Ross [1979] 15.5–15.13.

D.3. Character group:

Let G be a locally compact abelian group. A continuous group homomorphism χ from G into the unit circle Γ is called a character of G. The set of all characters of G is called the *character group* or *dual group* of G, denoted by $\widehat{G}$. Endowed with the pointwise multiplication and the compact-open topology G becomes a topological group which is commutative and locally compact (see Hewitt-Ross [1979], 23.15).

D.4 Proposition:

If G is a compact abelian group then $\widehat{G}$ is discrete; and if $\widehat{G}$ is a discrete abelian group, G is compact (see Hewitt-Ross [1979], 23.17).

D.5 Example: Let $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with multiplication and topology induced by $\mathbb{C}$. Then Γ is a compact group. Moreover, each character of Γ is of the form

$$z \mapsto z^n$$

for some $n \in \mathbb{Z}$, and therefore $\widehat{\Gamma}$ is isomorphic to $\mathbb{Z}$. Finally, the normalized Haar measure is the normalized one-dimensional Lebesgue measure m on Γ.

D.6. Pontrjagin's duality theorem:

Let G be a locally compact abelian group, and denote by $\widehat{\widehat{G}}$ the dual group of $\widehat{G}$.

$\hat{\hat{G}}$ is naturally isomorphic to G, where the isomorphism

$$\Phi : G \to \hat{\hat{G}}$$

is given by
$$g \mapsto \hat{\hat{g}} \quad \text{with } \hat{\hat{g}}(\chi) := \chi(g)$$

for all $\chi \in \hat{G}$ (see Hewitt-Ross [1979], 24.8).
In particular, this theorem asserts that a locally compact abelian group is uniquely determined by its dual.

D.7 Corollary:
The characters of a compact abelian group G form an orthonormal basis for $L^2(G, \mathcal{B}, m)$, $\mathcal{B}$ the Borel algebra and m the normalized Haar measure on G.

Proof. First, we prove the orthogonality by showing that $\int \chi(g)\, \mathrm{d}m(g) = 0$ for $\chi \neq 1$. Choose $h \in G$ with $\chi(h) \neq 1$. Then we have

$$\int \chi(g)\, \mathrm{d}m(g) = \int \chi(hg)\, \mathrm{d}m(g) = \chi(h) \int \chi(g)\, \mathrm{d}m(g)$$

and hence
$$\int \chi(g)\, \mathrm{d}m(g) = 0.$$

Clearly, every character is a normalized function in $L^2(G, \mathcal{B}, m)$. Let $g, h \in G$, and observe by (D.6) that there is a $\chi \in \hat{G}$ such that $\chi(g) \neq \chi(h)$, i.e. the characters separate the points of G. Therefore, the Stone-Weierstrass theorem implies that the algebra $\mathscr{A}$ generated by G, i.e. the vector space generated by $\hat{G}$, is dense in $C(G)$, and thus in $L^2(G, \mathcal{B}, m)$. ∎

We conclude this appendix with Kronecker's theorem which is useful for investigating rotations on the torus Γ^n. For elementary proofs see (III.8.iii) for $n = 1$ and Katznelson [1976], Ch. VI, 9.1 for general $n \in \mathbb{N}$. Our abstract proof follows Hewitt-Ross [1979], using duality theory.

D.8. Kronecker's theorem:
Let $a := (a_1, \dots, a_n) \in \Gamma^n$ be such that $\{a_, \dots, a_n\}$ linearly independent in the $\mathbb{Z}$-module Γ, i.e. $1 = a_1^{z_1} \dots a_n^{z_n}$, $z_i \in \mathbb{Z}$ implies $z_i = 0$ for $i = 1, \dots, n$. Then the subgroup $\{a^z : z \in \mathbb{Z}\}$ is dense in Γ^n.

Proof. Endow $\hat{\mathbb{Z}} = \Gamma$ with the discrete topology and form the dual group $\hat{\hat{\mathbb{Z}}}_d = \widehat{\Gamma_d}$. $\widehat{\Gamma_d}$ is a compact subgroup of the product Γ^Γ – note that here the compact-open topology on $\widehat{\Gamma_d}$ is the topology induced from the product Γ^Γ.
We consider the continuous monomorphism

$$\Phi : \mathbb{Z} \to \hat{\hat{\mathbb{Z}}}_d$$
$$z \mapsto \Phi(z) \quad \text{defined by } \Phi(z)(\gamma) := \gamma^z \text{ for all } \gamma \in \Gamma = \hat{\mathbb{Z}}.$$

Then the duality theorem yields that $\Phi(\mathbb{Z})$ is dense in $\hat{\hat{\mathbb{Z}}}_d$.
Now let $b := (b_1, \dots, b_n) \in \Gamma^n$ and $\varepsilon > 0$. Since $\{a_1, \dots, a_n\}$ is linearly independent in the $\mathbb{Z}$-module Γ there exists a $\mathbb{Z}$-linear mapping

$$\chi \in \widehat{\Gamma_d} \quad \text{with } \chi(a_i) = b_i \text{ for } i = 1, \dots, n.$$

By definition of the product topology on Γ^Γ and by denseness of $\Phi(\mathbb{Z})$ in $\widehat{\Gamma_d}$ we obtain $z \in \mathbb{Z}$ such that

$$\left|a_i^z - b_i\right| = \left|\Phi(z)(a_i) - \chi(a_i)\right| < \varepsilon,$$

for $i = 1, \ldots, n$. ■