Appendix C. Remarks on Banach Lattices and Commutative Banach Algebras

(i) Banach lattices

A large part of ergodic theory, as presented in our lectures, takes place in the concrete function spaces as introduced in (B.18)–(B.20). But these spaces bear more structure than simply that of a Banach space. Above all it seems to us to be the order structure of these function spaces and the positivity of the operators under consideration which is decisive for ergodic theory. For the abstract theory of Banach lattices and positive operators we refer to the monograph of H.H. Schaefer [1974] where many of the methods we apply in concrete cases are developed. Again, for the readers convenience we collect some of the fundamental examples, definitions and results.

C.1. Order structure on function spaces:
Let E be one of the real function spaces $C(X)$ or$L^p(X,\Sigma,\mu)$, $1 \leqslant p < \infty$. Then we can transfer the order structure of $\mathbb{R}$ to E in the following way:

For $f, g \in E$ we call f *positive*, denoted $f \geqslant 0$, if $f(x) \geqslant 0$ for all $x \in X$, and define

$f \vee g$, the *supremum* of f and g, by $(f \vee g)(x) := \sup\{f(x), g(x)\}$ for all $x \in X$

$f \wedge g$, the *infimum* of f and g, by $(f \wedge g)(x) := \inf\{f(x), g(x)\}$, for all $x \in X$

$|f|$, the absolute value of f, by $|f|(x) := |f(x)|$ for all $x \in X$.

The new functions $f \vee g$, $f \wedge g$ and $|f|$ again are elements of E.

Remark that for $E = L^p(X,\Sigma,\mu)$ the above definitions make sense either by considering representatives of the equivalence classes or by performing the operations for μ-almost all $x \in X$.

Using the positive cone $E_+ := \{f \in E : f \geqslant 0\}$ we define an order relation on E by $f \geqslant g$ if $(g - f) \in E_+$. Then E becomes an ordered vector space which is a lattice for $\vee$ and $\wedge$.
Moreover, the norm of E is compatible with the lattice structure in the sense that $0 \leqslant f \leqslant g$ implies $\|f\| \leqslant \|g\|$, and $\||f|\| = \|f\|$ for every $f \in E$.

If we consider a *complex* function space E then the order relation "$\leqslant$" is defined only on the real part E_r consisting of all real valued functions in E. But the absolute value $|f|$ makes sense for all $f \in E$, and $\||f|\| = \|f\|$ holds.

C.2. A *Banach lattice* E is a real Banach space endowed with a vector ordering "$\leqslant$" making it into a vector lattice (i.e. $|f| = f \vee (-f)$ exists for every $f \in E$ and satisfying the compatibility condition:

$$|f| \leqslant g \quad \text{implies} \quad \|f\| \leqslant \|g\| \quad \text{for all } fg \in E.$$

Complex Banach lattices can be defined in a canonical way analogous to the complex function spaces in (C.1) (see Schaefer [1974], Ch.II,§11).

C.3. Let E be a Banach lattice. A subset A of E is called *order bounded* if A is contained in some *order interval* $[g, h] := \{f \in E : g \leqslant f \leqslant h\}$ for $g, h \in E$. The Banach lattice E is order complete if for every order bounded subset A the supremum $\sup A$ exists. Examples of order complete Banach lattices are the spaces $L^p(\mu)$, $1 \leqslant p \leqslant \infty$, while $C([0,1])$ is not order complete.

C.4. Positive operators:
Let E, F be (real or complex) Banach lattices and $T : E \to F$ a continuous linear operator. T is *positive* if $TE_+ \subseteq F_+$, or equivalently, if $T|f| \geqslant |Tf|$ for all $f \in E$.

The morphisms for the vector lattice structure, called *lattice homomorphisms*, satisfy the stronger condition $T|f| = |Tf|$ for every $f \in E$.

If the norm on E is strictly monotone (i.e. $0 \leqslant f < g$ implies $\|f\| < \|g\|$); e.g. $E = L^p(\mu)$ for $1 \leqslant p < \infty$) then every positive isometry T on E is a lattice homomorphism. In fact, in that case $|Tf| \leqslant T|f|$ and $\||Tf|\| = \|Tf\| = \|f\| = \||f|\| = \|T|f|\|$ imply $|Tf| = T|f|$.

Finally, T is called *order continuous* (countably order continuous) if $\inf_{\alpha \ni A} Tx_\alpha = 0$ for every downward directed net (sequence) $(x_\alpha)_{\alpha \in A}$ with $\inf_{\alpha \ni A} x_\alpha = 0$.

C.5. Examples of positive operators are provided by positive matrices and integral operators with positive kernel (see Schaefer [1974], Ch. IV, §8).

Further, the multiplication operator

$$M_g : C(X) \to C(X) \quad (\text{resp. } L^p(X, \Sigma, \mu) \to L^p(X, \Sigma, \mu))$$

is a lattice homomorphism for every $0 \leqslant g \in C(X)$ (resp. $0 \leqslant g \in L^\infty(X, \Sigma, \mu)$).
The operators

$$T_\varphi : f \mapsto f \circ \varphi$$

induced in $C(X)$ or $L^p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, by suitable transformations

$$\varphi : X \to X$$

are even lattice homomorphisms (see II.4).

(ii) Commutative Banach algebras

While certainly order and positivity are more important for ergodic theory, in some places we use the multiplicative structure of certain function spaces.

C.6. Algebra structure on function spaces:
Let E be one of the complex function spaces $C(X)$ or $L^\infty(X, \Sigma, \mu)$. Then the multiplicative structure of $\mathbb{R}$ can be transferred to E: for $f, g \in E$ we define

$f \cdot g$, the *product* of f and g, by $(f - g)(x) := f(x) \cdot g(x)$ for all $x \in X$,

f^*, the *adjoint* of f, by $f^*(x) := \overline{f(x)}$ for all $x \in X$ where "—" denotes the complex conjugation.
The function R$\mathbf{1}$, defined by $\mathbf{1}(x) := 1$ for all $x \in X$, is the neutral element of the above commutative multiplication. The operation "$*$" is an *involution.*

C.7. A *C^*-algebra* $\mathscr{A}$ is a complex Banach space and an algebra with involution $*$ satisfying

$$\|f \cdot f^*\| = \|f\|^2$$

for all $f \in \mathscr{A}$.
For our purposes we may restrict our attention to commutative C^*-algebras. As shown in (C.6) the function spaces $C(X)$ and $L^\infty(X, \Sigma, \mu)$ are commutative C^*-algebras. Another example is the sequence space ℓ^∞.

C.8. Multiplicative operators:
Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two C^*-algebras. The morphisms

$$T : \mathscr{A}_1 \to \mathscr{A}_2$$

corresponding to the C^*-algebra structure of $\mathscr{A}_1$ and $\mathscr{A}_2$ are continuous linear operators satisfying

$$T(f \cdot g) = (Tf) \cdot (Tg)$$

and

$$T(f^*) = (Tf)^*$$

for all $f, g \in \mathscr{A}$.

Let $\mathscr{A} = C(X)$, resp. $L^\infty(X, \Sigma, \mu)$. If $\varphi : X \to X$ is a continuous, resp. measurable, transformation, the induced operator

$$T_\varphi : f \mapsto f \circ \varphi$$

is a *multiplicative* operator on $\mathscr{A}$ satisfying $T_\varphi \mathbf{1} = \mathbf{1}$ and $T_\varphi f^* = (T_\varphi f)^a st$ (see II.4).

C.9. Representation theorem of Gelfand-Neumark:
Every commutative C^*-algebra $\mathscr{A}$ with unit is isomorphic to a space $C(X)$. Here X may be identified with the set of all non-zero multiplicative linear forms on $\mathscr{A}$, endowed with the weak* topology (see Sakai [1971], 1.2.1).

We remark that for $\mathscr{A} = \ell^\infty(\mathbb{N})$ the space X is homeomorphic to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ (see Schaefer [1974], p. 106), and for $\mathscr{A} = L^\infty(Y, \Sigma, \mu)$, X may be identified with the Stone representation space of the measure algebra $\check{\Sigma}$ (see VI.D.6).