Appendix B. Some Functional Analysis

As indicated in the introduction, the present lectures on ergodic theory require some familiarity with functional-analytic concepts and with functional-analytic thinking. In particular, properties of Banach spaces E, their duals E' and the bounded linear operators on E and E' play a central role. It is impossible to introduce the newcomer into this world of Banach spaces in a short appendix. Nevertheless, in a short "tour d'horizon" we put together some more or less standard definitions, arguments and examples – not as an introduction into functional analysis but as a reminder of things you (should) already know or as a reference of results we use throughout the book. Our standard source is Schaefer [1971].

B.1. Banach spaces:
Let E be a real or complex *Banach space* with norm $\|\cdot\|$ and closed unit ball $U := \{f \in E : \|f\| \leqslant 1\}$. We associate to E its dual E' consisting of all continuous linear functionals on E. Usually, E' will be endowed with the dual norm

$$\|f'\| := \sup\{|\langle f, f'\rangle| : \|f\| \leqslant 1\}$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical bilinear form

$$(f, f') \mapsto \langle f, f'\rangle := f'(f) \quad \text{on } E \times E'.$$

B.2. Weak topologies:
The topology on E of pointwise convergence on E' is called the weak topology and will be denoted by $\sigma(E, E')$. Analogously, one defines on E' the topology of pointwise convergence on E, called the weak* topology and denoted by $\sigma(E', E)$. These topologies are weaker than the corresponding strong (= norm) topologies, and we need the following properties.

B.3. While in general not every strongly closed subset of a Banach space E is weakly closed, it is true that the strong and weak closure coincide for convex sets (Schaefer [1971], II.9.2, Corollary 2).

B.4. Theorem Alaoglu-Bourbaki:
The dual unit ball $U^\circ := \{f' \in E' : \|f'\| \leqslant 1\}$ in E' is weak* compact (Schaefer [1971], IV.5.2).

From this one deduces: A Banach space E is reflexive (i.e. the canonical injection from E into the bidual E'' is surjective) if and only if its unit ball is weakly compact (Schaefer [1971], IV.5.6).

B.5. Theorem of Krein-Milman
Every weak* compact, convex subset of E' is the closed, convex hull of its set of extreme points (Schaefer [1971], II.10.4).

B.6. Theorem of Krein:
The closed, convex hull of a weakly compact set is still weakly compact (Schaefer [1971], IV. 11.4).

B.7. Bounded operators:
Let T be a bounded (=continuous) linear operator on the Banach space E. Then T is called a *contraction* if $\|Tf\| \leqslant \|f\|$, and an *isometry* if $\|Tf\| = \|f\|$ for all $f \in E$. We remark that every bounded linear operator T on E is automatically continuous for the weak topology on E (Schaefer [1971], III.1.1). For $f \in E$ and $f' \in E'$ we define the corresponding *one-dimensional operator*

$$f' \otimes f \quad \text{by} \quad (f' \otimes f)(g) := \langle g, f' \rangle \cdot f$$

for all $g \in E$. Moreover we call a bounded linear operator P on E a *projection* if $P^2 = P$. In that case we have $P^2 = P$.

Proposition: For a projection P on a Banach space E the dual of PE is (as a topological vector space) isomorphic to the closed subspace $P'E'$ of E'.

Proof. The linear map $\Phi : E' \to (PE)'$ defined by $\Phi f' := f|_{PE}$ is surjective by the Hahn-Banach theorem. Therefore $(PE)'$ is isomorphic to $E'/\ker\Phi$. From $\ker\Phi = {P'}^{-1}(0)$ and $E' = P'E' \oplus {P'}^{-1}(0)$ we obtain $(PE)' \simeq E'/{P'}^{-1}(0) \simeq P'E'$. ∎

B.8. The space $\mathscr{L}(E)$ of all bounded linear operators on E becomes a Banach space if endowed with the *operator norm*

$$\|T\| := \sup\{\|Tf\| : \|f\| \leqslant 1\}.$$

But other topologies on $\mathscr{L}(E)$ will be used as well. We write $\mathscr{L}_s(E)$ if we endow $\mathscr{L}(E)$ with the *strong operator topology* i.e. with the topology of simple (= pointwise) convergence on E with respect to the norm topology. Therefore, a net $\{T_\alpha\}$ converges to T in the strong operator topology iff $T_\alpha \xrightarrow{\|\cdot\|} Tf$ for all $f \in E$. Observe that the strong operator topology is the topology on $\mathscr{L}(E)$ induced from the product topology on $(E, \|\cdot\|)^E$.

The *weak operator topology* on $\mathscr{L}(E)$ – write $\mathscr{L}_w(E)$ – is the topology of simple convergence on E with respect to $\sigma(E, E')$. Therefore,

$$\begin{aligned} &T_\alpha \text{ converges to } T \text{ in the weak operator topology} \\ \text{iff} \quad &\langle T_\alpha f, f' \rangle \to \langle Tf, f' \rangle \quad \text{for all } f \in E,\ f' \in E'. \end{aligned}$$

Again, this topology is the topology on $\mathscr{L}(E)$ inherited from the product topology on $(E, \sigma(E, E'))^E$.

B.9. Bounded subsets of $\mathscr{L}(E)$**:**
For $M \subseteq \mathscr{L}(E)$ the following are equivalent:

(a) M is bounded for the weak operator topology.
(b) M is bounded for the strong operator topology.
(c) M is uniformly bounded, i.e. $\sup\{\|T\| : T \in M\} < \infty$.
(d) M is equicontinuous for $\|\cdot\|$.

Proof. See Schaefer [1971], III.4.1, Corollary, and III.4.2 for (b) ⇔ (c) ⇔ (d); for (a) ⇔ (b) observe that the duals $\mathscr{L}_s(E)$ and $\mathscr{L}_w(E)$ are identical (Schaefer [1971], IV.4.3, Corollary 4). Consequently, the bounded subsets agree (Schaefer [1971], IV.3.2, Corollary 2). ∎

B.10. If M is a bounded subset of $\mathscr{L}(E)$, then the closure of M as subset of the product$(E, \|\cdot\|)^E$ is still contained in $\mathscr{L}(E)$ (Schaefer [1971], III.4.3).

B.11. On bounded subsets M of $\mathscr{L}(E)$, the topology of pointwise convergence on a total subset A of E coincides with the strong operator topology. Here we call A "total" if its linear hull is dense in E (Schaefer [1971], III.4.5).

The advantage of the strong, resp. weak, operator topology versus the norm topology on $\mathscr{L}(E)$ is that more subsets of $\mathscr{L}(E)$ become compact. Therefore, the following assertions (B.12)–(B.15) are of great importance.

B.12 Proposition:
For $M \subseteq \mathscr{L}(E)$, $g \in E$, we define the orbit $Mg : \{Tg : T \in M\} \subseteq E$, and the

subspaces $\quad G_s := \{f \in E : Mf \text{ is relatively } \|\cdot\|\text{-compact}\}$

and $\quad G_\sigma := \{f \in E : Mf \text{ is relatively } \sigma(E,E')\text{-compact}\}.$

If M is bounded, then G_s and G_σ are $\|\cdot\|$-closed in E

Proof. The assertion for G_s follows by a standard diagonal procedure. The argument for G_σ is more complicated: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in G_σ converging to $f \in E$. By the theorem of Eberlein (Schaefer [1971], IV.11.2) it suffices to show that every sequence $(T_k f)_{k\in\mathbb{N}}$, $T_k \in M$ has a subsequence which converges weakly. Since $f_1 \in G_\sigma$ there is a subsequence $(T_{k_{i_1}} f_1)$ weakly converging to some $g_1 \in E$. Since $f_2 \in G_\sigma$, there exists a subsequence such that $(T_{k_{i_2}} f_2)$ such weakly converges to g_2, and so on. Applying a diagonal procedure we find a subsequence $(T_{k_i})_{i\in\mathbb{N}}$ of $(T_k)_{k\in\mathbb{N}}$ such that $T_{k_i} f_n \overset{i\to\infty}{\longrightarrow} g_n \in E$ weakly for every $n \in \mathbb{N}$. From

$$\begin{aligned}\|g_n - g_m\| &= \sup\big\{\langle g_n - g_m, f'\rangle : \|f'\| \leqslant 1\big\}\\ &= \sup\big\{\lim_{i\to\infty} |\langle T_{k_i} f_n - T_{k_i} f_m, f'\rangle| : \|f'\| \leqslant 1\big\}\\ &\leqslant \|T_{k_i}\| \cdot \|f_n - f_m\|\end{aligned}$$

it follows that $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and therefore converges to some $g \in E$. A standard 3ε-argument shows $T_{k_i} f \overset{i\to\infty}{\longrightarrow} g$ for $\sigma(E,E')$. ∎

B.13 Proposition:
For a bounded subset $M \subseteq \mathscr{L}(E)$ the following are equivalent:

(a) M is relatively compact for the strong operator topology.
(b) Mf is relatively compact in E for every $f \in E$.
(c) Mf is relatively compact for every f in a total subset of E.

Proof. (a) ⇒ (b) follows by the continuity of the mapping $T \mapsto Tf$ from $\mathscr{L}_s(E)$ into E.

(b) ⇔ (c) follows from (B.12), and (c) ⇒ (a) is a consequence of (A.3) and (B.10). ∎

B.14 Proposition:
For a bounded subset $M \subseteq \mathscr{L}(E)$ the following are equivalent:

(a) M is relatively compact for the weak operator topology.
(b) Mf is relatively weakly compact for every $f \in E$.
(c) Mf is relatively weakly compact for every f in a total subset of E.

The proof follows as in (B.13).

B.15 Proposition:
Let $M \subseteq \mathscr{L}(E)$ be compact and choose a total subset $A \subseteq E$ and a $\sigma(E', E)$-total subset $A \subseteq E'$. Then the weak operator topology on M coincides with the topology of pointwise convergence on A and A'. In particular, M is metrizable if E is separable and E' is $\sigma(E', E)$-separable ("separable" means that there exists a countable dense set).

Proof. The semi-norms

$$P_{f,f'}(T) := |\langle Tf, f' \rangle|, \quad T \in M, f \in A, f' \in A'$$

define a Hausdorff topology on M coarser than the weak operator topology. Since M is compact, both topologies coincide (see A.2). ∎

B.16. Continuity of the multiplication in $\mathscr{L}(E)$:
In Lecture VII the *multiplication*

$$(S, T) \mapsto S \circ T$$

in $\mathscr{L}(E)$ plays an important role. Therefore, we state its continuity properties: The multiplication is jointly continuous on $\mathscr{L}(E)$ for the norm topology. In general, it is only separately continuous for the strong or the weak operator topology. However, it is jointly continuous on bounded subsets of $\mathscr{L}_s(E)$ (see Schaefer [1971], p. 183).

B.17. Spectral theory:
Let E be a complex Banach space and $T \in \mathscr{L}(E)$. The *resolvent set* $\rho(T)$ consists of all complex numbers λ for which the *resolvent* $R(\lambda, T) := (\lambda - T)^{-1}$ exists. The mapping $\lambda \mapsto R(\lambda, T)$ is holomorphic on $\rho(T)$. The *spectrum* $\sigma(T) := \mathbb{C} \backslash \rho(T)$ is a non-empty compact subset of $\mathbb{C}$, and two subsets of $\sigma(T)$ are of special interest: the *point spectrum*

$$P\sigma(T) := \{\lambda \in \sigma(T) : (\lambda - T) \text{ is not injective}\}$$

and the *approximate point spectrum*

$$A\sigma(T) := \{\lambda \imath \sigma(T) : (\lambda - T) f_n \to 0 \text{ for some normalized sequence } (f_n)\}.$$

A complex number λ is called an (approximate) eigenvalue if $\lambda \in P\sigma(A)$ (resp. $\lambda \in A\sigma(T)$), and $F_\lambda := \{f \in E : (\lambda - T) = 0\}$ is the eigenspace corresponding to the eigenvalue λ; λ is a simple eigenvalue if $\dim F_\lambda = 1$.

The real number $r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}$ is called the *spectral radius* of T, and may be computed from the formula $r(T) = \lim_{n \to \infty} \left(\|T^n\|\right)^{\frac{1}{n}}$.

If $|\lambda| > r(T)$ the resolvent can be expressed by the Neumann series

$$R(\lambda, T) = \sum_{n=0}^{\infty} \lambda^{(n+1)} T^n.$$

For more information we refer to Schaefer [1971], App. 1 and Reed-Simon [1972].

B.18. The spaces $C(X)$ and their duals $M(X)$:
Let X be a compact space. The space $C(X)$ of all real (resp. complex) valued continuous functions on X becomes a Banach space if endowed with the norm

$$\|f\| := \sup\{|f(x)| : x \in X\}, \quad f \in C(X).$$

The dual of $C(X)$, denoted $M(X)$, is called the space of Radon measures on X. By the theorem of Riesz (Bauer [1972], 7.5) $M(X)$ is (isomorphic to) the set of all regular real-(resp. complex-)valued Borel measures on X (see A.12).

The Dirac measures δ_x, $x \in X$, defined by $\langle \delta_x, f\rangle := f(x)$ for all $f \in C(X)$, are elements of $M(X)$, and we obtain from Lebesgue's dominated convergence theorem (see A.16) the following:

If $f_n, f \in C(X)$ with $\|f_n\| \leqslant c$ for all $n \in \mathbb{N}$, then f_n converges to f for $\sigma(C(X), M(X))$ if and only if $\langle f_n, \delta_x\rangle \to \langle f, \delta_x\rangle$ for all $x \in X$.

B.19. Sequence spaces:
Let D be a set and take $1 \leqslant p < \infty$. The sequence space $\ell^p(D)$ is defined by

$$\ell^p(D) := \Big\{(x_d)_{d\in D} : \sum_{d\in D} |x_d|^p < \infty\Big\}$$

where x_D are real (or complex) numbers.

Analogously, we define

$$\ell^\infty(D) := \Big\{(x_d)_{d\in D} : \sup_{d\in D} |x_d| < \infty\Big\}.$$

The vector space $\ell^p(D)$, resp. $\ell^\infty(D)$, becomes a Banach space if endowed with the norm

$$\|(x_d)_{d\in D}\| := \Big(\sum_{d\in D} |x_d|^p\Big)^{1/p},$$

resp.

$$\|(x_d)_{d\in D}\| := \sup_{d\in D} |x_d|.$$

In our lectures, D equals $\mathbb{N}$, $\mathbb{N}_0$ or $\mathbb{Z}$. Instead of $\ell^p(D)$ we write ℓ^p if no confusion is possible.

B.20. The $L^p(X, \Sigma, \mu)$:
Let (X, Σ, μ) be a measure space and take $1 \leqslant p < \infty$. By $\mathscr{L}(X, \Sigma, \mu)$ we denote the vector space of all real- or complex-valued measurable functions on X with $\int_X |f|^p \, d\mu < \infty$. Then

$$\|f\|_p := \Big(\int_X |f|^p \, d\mu\Big)^{1/p}$$

is a semi-norm on $\mathscr{L}^p(X, \Sigma, \mu)$,

$$N_\mu := \big\{f \in \mathscr{L}^p(X, \Sigma, \mu) : \|f\|_p = 0\big\}$$

and is a closed subspace. The quotient space

$$L^p(X, \Sigma, \mu) = L^p(\mu) := \mathscr{L}^p(X, \Sigma, \mu)/N_\mu$$

endowed with the quotient norm is a Banach space. Analogously, one denotes by $\mathscr{L}^\infty(X, \Sigma, \mu)$ the vector space of μ-essentially bounded measurable functions on X. Again,

$$\|f\|_\infty := \big\{c \in \mathbb{R}^+ : \mu[|f| > c] = 0\big\}$$

yields a semi-norm on $\mathscr{L}^\infty(X,\Sigma,\mu)$ and the subspace

$$N_\mu := \left\{f \in \mathscr{L}^\infty(X,\Sigma,\mu) : \|f\|_\infty = 0\right\}$$

is closed. The quotient space

$$L^\infty(X,\Sigma,\mu) = L^\infty(\mu) := \mathscr{L}^\infty(X,\Sigma,\mu)/N_\mu$$

is a Banach space.

Even if the elements of $L^p(X,\Sigma,\mu)$ are equivalence classes of functions it generally causes no confusion if we calculate with the function $f \in \mathscr{L}^p(X,\Sigma,\mu)$ instead of its equivalence class $\check{f} \in L^p(X,\Sigma,\mu)$ (see II.D.4).

In addition, most operators used in ergodic theory are initially defined on the spaces $\mathscr{L}^p(X,\Sigma,\mu)$. However, if they leave invariant N_μ, we can and shall consider the induced operators on $L^p(X,\Sigma,\mu)$

B.21. For $1 \leqslant p < \infty$ the Banach space $L^p(X,\Sigma,\mu)$ is separable if and only if the measure algebra $\check{\Sigma}$ is separable.

B.22. If the measure space (X,Σ,μ) is finite, then

$$L^\infty(\mu) \subseteq L^{p_2}(\mu) \subseteq L^{p_1}(\mu) \subseteq L^1(\mu)$$

for $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$.

B.23. Let (X,Σ,μ) be σ-finite. Then the dual of $L^p(X,\Sigma,\mu)$, $1 < p < \infty$ is isomorphic to $L^q(X,\Sigma,\mu)$ where $\frac{1}{p}+\frac{1}{q}=1$, and the canonical bilinear form is given by

$$\langle f,g\rangle = \int f\cdot g \,\mathrm{d}\mu \quad \text{for } f \in L^p(\mu),\ g \in L^1(\mu).$$

Analogously, the dual of $L^1(\mu)$ is isomorphic to $L^\infty(\mu)$.

B.24. Conditional expectation:
Given a measure space (X,Σ,μ) and a sub-σ-algebra $\Sigma_0 \subseteq \Sigma$, we denote by J the canonical injection from $L^p(X,\Sigma_0,\mu)$ into $L^p(X,\Sigma,\mu)$ for $1 \leqslant p \leqslant \infty$. J is contractive and positive (see C.4). Its (pre-)adjoint

$$P : L^q(X,\Sigma,\mu) \to L^q(X,\Sigma_0,\mu)$$

is a positive contractive projection satisfying

$$P(f\cdot g) = g\cdot P(f) \quad \text{for } f \in L^q(X,\Sigma,\mu),\ g \in L^\infty(X,\Sigma_0,\mu).$$

Proof. P is positive and contractive since J enjoys the same properties. The above identity follows from

$$\langle P(fg),h\rangle = \langle fg,Jh\rangle = \int fgh\,\mathrm{d}\mu = \langle f,J(gh)\rangle\rangle = \langle (Pf)g,h\rangle$$

for all (real) $h \in L^p(X,\Sigma_0,\mu)$. ∎

We call P the *conditional expectation operator* corresponding to Σ_0. For its probabilistic interpretation see Ash [1972], Ch. 6.

B.25. Direct sums:
Let E_i, $i \in \mathbb{N}$, be Banach spaces with corresponding norms $\|\cdot\|_i$, and let $1 \leqslant p < \infty$. The ℓ^p*-direct sum* of $(E_i)_{i\in\mathbb{N}}$ is defined by

$$E := \bigoplus_p E_i := \Big\{(x_i)_{i\in\mathbb{N}} : x_i \in E_i \text{ for all } i \in \mathbb{N} \text{ and } \textstyle\sum_{i\in\mathbb{N}} \|x_i\|_i^p < \infty\Big\}.$$

E is a Banach space under the norm

$$\|(x_i)_{i\in\mathbb{N}}\| := \Big(\sum_{i\in\mathbb{N}} \|x_i\|_i^p\Big)^{1/p}$$

Given $S_i \in \mathscr{L}(E_i)$ with $\sup_{i\in\mathbb{N}} \|S_i\| < \infty$, then

$$\bigoplus S_i : (x_i)_{i\in\mathbb{N}} \mapsto (S_i x_i)_{i\in\mathbb{N}}$$

is a bounded linear operator on E with $\|\bigoplus S_i\| = \sup\{\|S_i\| : i \in \mathbb{N}\}$. Analogously one defines the ℓ^∞-direct sum $\bigoplus_\infty E_i$.