Prof. Dr. H. D. Alber

WS 03/04
01. 02. 08

AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Solutions to the Exercises of Tutorial 14

1. *Proof.* Let $\Omega_1,\ \Omega_2$ be bounded open sets of $\mathbb{R}^n$ with $n \in \mathbb{N}$ satisfying

$$\Omega_1 \subset \Omega_2.$$

We want to prove that an function f in $\overset{\circ}{C}_\infty(\Omega_1)$ can be extended, by 0, to $\hat{f} \in \overset{\circ}{C}_\infty(\Omega_2)$. To this end, we choose $f \in \overset{\circ}{C}_\infty(\Omega_1)$. This mean f vanished near the boundary of Ω_1 say $f(x) = 0$ for any $x \in \Omega_1^\delta$, where $\Omega_1^\delta = \{x \in \Omega_1 \mid \mathrm{dist}(x, \partial\Omega_1) \leq \delta\}$, for suitably small positive δ. Thus if we define $\hat{f}(x) = 0$ for any $x \in \Omega_1^\delta \cup (\Omega_2 \setminus \Omega_1)$, then one can easily prove that $\hat{f} \in \overset{\circ}{C}_\infty(\Omega_2)$.

Recalling that $\overset{\circ}{H}_1(\Omega_1)$ is the closure of $\overset{\circ}{C}_\infty(\Omega)$ in the $H_1(\Omega)$-norm, we conclude that for any $u \in \overset{\circ}{H}_1(\Omega_1)$, define a function $\hat{u} : \Omega_2 \to \mathbb{C}$ by

$$\hat{u}(x) = \begin{cases} u(x), & x \in \Omega_1, \\ 0, & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

then we have that $\hat{u}$ belongs to $\overset{\circ}{H}_1(\Omega_2)$.

2. *Proof.* Define $B(u, v) = (\nabla_x u, \nabla_x v)$. Invoking the conclusion in Problem 1, we obtain

$$\begin{aligned} \lambda_1(\Omega_2) &= \min_{u \in \overset{\circ}{H}_1(\Omega_2), \|u\|_{L_2(\Omega_2)}=1} B(u, u) \\ &\leq \min_{u \in \overset{\circ}{H}_1(\Omega_1), \|u\|_{L_2(\Omega_1)}=1} B(u, u) \\ &= \lambda_1(\Omega_1). \end{aligned} \tag{1}$$

Here we used the basic inequality that

$$\min_{x \in B} f(x) \leq \min_{x \in A} f(x)$$

for $f : X \to \mathbb{R}$ and $A, B \subset X$ such that $A \subset B$. Here X is a Banach space. We here also used the assertion in Problem 1 which can be stated as follows:

"by zero extension we can regard $\overset{\circ}{H}_1(\Omega_1)$ as a subset of $\overset{\circ}{H}_1(\Omega_2)$".

By a similar argument for the formula similar to 1) we can prove that for any $k \in \mathbb{N} \setminus \{1\}$, there holds

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1).$$

3. and 4. *Proof.* Noting the main difference between the Neumann and Dirichlet boundary value problems: The test function space for the Dirichlet boundary value problem $\overset{\circ}{C}_\infty(\Omega)$ (or $\overset{\circ}{H}_1(\Omega)$) while for the Neumann boundary value problem the test space should be changed to $C_\infty(\Omega)$ (or $H_1(\Omega)$). We then can prove in a similar way as in the lecture, Lemma 9.2, Theorems 9.3, 9.4, Corollary 9.5, Theorem 9.6 and Corollaries 9.7, 9.8, for the operator of Neumann boundary value problem.