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AG 06, FB Mathematik
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Partial Differential Equations I: Linear Theory

Solutions to the Exercises of Tutorial 13

1. *Solution.* Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ and smooth boundary be a bounded open set and $f : \Omega \to \mathbb{R}$ be a function in $L_2(\Omega)$ and $u^{(b)} \in H_1(\Omega)$

A function u in the space $H_1(\Omega) \cap L_6(\Omega)$ is called a weak solution to

$$\begin{aligned} \Delta u - u^3 &= f, \text{ in } \Omega, \\ u|_{\partial\Omega} &= u^{(b)}, \end{aligned}$$

if for any test function $\varphi \in \overset{\circ}{C}_\infty(\Omega)$ there holds

$$(\nabla_x u, \nabla_x \varphi) + (u^3, \varphi) = -(f, \varphi), \tag{1}$$

where $u - u^{(b)} \in \overset{\circ}{H}_1(\Omega)$.

2. *Solution.* i) Let $\xi \in \mathbb{R}^{n+1}$. Then we have that

$$\xi_0^2 - \sum_{i=1}^{n} \xi_i^2 = 0$$

has non-zero solutions. Thus the operator

$$L_1 u(x_1, \cdots, x_n, t) = \frac{\partial^2}{\partial t^2} u - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u$$

is not elliptic. In fact, this is called *hyperbolic* operator.

ii) Let $\xi \in \mathbb{R}^{n+1}$. For the operator

$$L_2 u(x_1, \cdots, x_n, t) = \frac{\partial}{\partial t} u - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u,$$

we have

$$\sum_{i=1}^{n} \xi_i^2 = 0$$

has the solutions of the form $(c, 0, \cdots, 0)$ with $c \in \mathbb{R}$. So this operator is not elliptic. It is called parabolic operator.

iii) Let $\xi \in \mathbb{R}^2$. Then for the operator

$$L_3 u(x,y) = \frac{\partial^2}{\partial x^2}u + 2i\frac{\partial}{\partial x}u\frac{\partial}{\partial y}u - \frac{\partial^2}{\partial y^2}u,$$

we have that

$$\xi_1^2 + 2i\xi_1\xi_2 - \xi_2^2 = 0$$

implies

$$(\xi_1 + i\xi_2)^2 = 0,$$

thus $\xi_1 + i\xi_2 = 0$, i.e.

$$\xi_1 = 0, \quad \xi_2 = 0.$$

Therefore, this operator is elliptic. Moreover it is easy to see that this operator is not strongly or uniformly elliptic.

iv) Let $\xi \in \mathbb{R}^2$. We rewrite the operator as

$$\begin{aligned} L_4 u(x,y) &= \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u\right) + \frac{\partial^2}{\partial y^2}u \\ &= \frac{\partial}{\partial x}u + x\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u. \end{aligned}$$

Thus from

$$x\xi_1^2 + \xi_2^2 = 0$$

we obtain that if $x > 0$ i.e. $x \in \Omega_1$ L_4 is elliptic, but not uniformly, strongly.

For $x \in \Omega_2$, if $x > 0$ then L_4 is elliptic, if $x = 0$ then L_4 is degenerate elliptic, if $x < 0$ then L_4 is hyperbolic. We also call that L_4 in Ω_2 is an operator of mixed type.

3. *Proof.* Since

$$a_{\alpha\beta} = \overline{a_{\beta\alpha}},$$

we have

$$a_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} + \overline{a_{\beta\alpha}}).$$

Then

$$\begin{aligned} Lu(x) &= \sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(a_{\alpha\beta}D^\beta u(x)\right) \\ &= \sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(\frac{1}{2}(a_{\alpha\beta} + \overline{a_{\beta\alpha}})D^\beta u(x)\right) \\ &= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(a_{\alpha\beta}D^\beta u(x)\right) + \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(\overline{a_{\beta\alpha}}D^\beta u(x)\right) \\ &= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(a_{\alpha\beta}D^\beta u(x)\right) + \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^\beta\left(\overline{a_{\alpha\beta}}D^\alpha u(x)\right). \qquad (2) \end{aligned}$$

Since we assume that $a_{\alpha\beta} = \text{const}$, the last term in (2) can be treated as

$$\begin{aligned}
\frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^{\beta}\left(\overline{a_{\alpha\beta}}D^{\alpha}u(x)\right) &= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1}\left(\overline{a_{\alpha\beta}}D^{\beta}D^{\alpha}u(x)\right)\\
&= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1}\left(\overline{a_{\alpha\beta}}D^{\alpha}D^{\beta}u(x)\right)\\
&= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(\overline{a_{\alpha\beta}}D^{\beta}u(x)\right). \qquad (3)
\end{aligned}$$

Here we used $D^{\beta}D^{\alpha}u = D^{\alpha}D^{\beta}u$ which is true by assuming that $u \in C_2$ or understanding it in weak sense.

Thus combination of (2) and (3) yields

$$\begin{aligned}
Lu(x) &= \sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(a_{\alpha\beta}D^{\beta}u(x)\right)\\
&= \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(a_{\alpha\beta}D^{\beta}u(x)\right) + \frac{1}{2}\sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(\overline{a_{\alpha\beta}}D^{\beta}u(x)\right)\\
&= \sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(\frac{1}{2}\left(a_{\alpha\beta} + \overline{a_{\alpha\beta}}\right)u(x)\right)\\
&= \sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(c_{\alpha\beta}D^{\beta}u(x)\right),
\end{aligned}$$

here we chose

$$c_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} + \overline{a_{\alpha\beta}}\right),$$

from which one can easily see that $c_{\alpha\beta} \in \mathbb{R}$. And the proof of the assertion is thus complete.

4. *Solution.* We define

$$a_{\alpha\beta} = x + iy$$

if $\alpha = (a_1, a_2)$ and $\beta = (b_1, b_2)$ where $a_i, b_i \in \{0, 1\}$ for $i = 1, 2$ satisfying that the four numbers a_1, a_2, b_1, b_2 are not equal pairwise; and define

$$a_{\alpha\beta} = 0,$$

otherwise.

Then we can write the operator

$$Lu(x,y) = \frac{\partial}{\partial x}\left((x+iy)\frac{\partial}{\partial y}u(x,y)\right) + \frac{\partial}{\partial y}\left((x+iy)\frac{\partial}{\partial x}u(x,y)\right)$$

in the form

$$Lu(x,y) = \sum_{|\alpha|=1,|\beta|=1} D^{\alpha}\left(a_{\alpha\beta}(x,y)D^{\beta}u(x,y)\right).$$

Now we take $\alpha = (1,0)$ and $\beta = (0,1)$. Then $a_{\alpha\beta} = x + iy$ and $\alpha + \beta = (1,1)$ whence $|\alpha + \beta| = 2$. Therefore, for such α, β, the symmetry condition becomes

$$x + iy = a_{\alpha\beta} = (-1)^{|\alpha+\beta|}\overline{a_{\beta\alpha}}(x,y) = \overline{a_{\beta\alpha}}(x,y) = \overline{x + iy}.$$

This can not be true for all $y \neq 0$.

Thus the symmetry conditions are not all satisfied.