Prof. Dr. H.-D. Alber

WS 07/08
18. 01. 08

AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory

Solutions to the Exercises of Tutorial 12

1. *Proof.* i) Suppose that $\|K\| = 0$. We want to show that $K = 0$. By definition, we have

$$0 = \|K\| \geq \frac{\|Kx\|}{\|x\|},$$

for all $x \in X$, whence $\|Kx\| = 0$, so $Kx = 0$ for all $x \in X$. This means $K = 0$.

ii) It is easy to see that $\|\lambda K\| = |\lambda| \, \|\lambda K\|$ for all $\lambda \in \mathbb{R}$.

iii) The triangle inequality. For any two operators K, L, we have

$$\begin{aligned} \|K + L\| &= \sup_{x \in X, x \neq 0} \frac{\|(K+L)x\|}{\|x\|} \\ &\leq \sup_{x \in X, x \neq 0} \frac{\|Kx\| + \|Lx\|}{\|x\|} \\ &\leq \sup_{x \in X, x \neq 0} \frac{\|Kx\|}{\|x\|} + \sup_{x \in X, x \neq 0} \frac{\|Lx\|}{\|x\|} \\ &= \|K\| + \|L\|. \end{aligned}$$

So $\|K\|$ is a norm of K.

2. *Proof.* Choose a Cauchy sequence in $\mathcal{B}(X, X)$, say $\{K_n\}_n$, which satisfies

$$\|K_n - K_m\| \to 0, \tag{1}$$

as $n, m \to \infty$.

Define a sequence

$$y_n = K_n x$$

for any $x \in X$. By triangle inequality we see that

$$\|y_n - y_m\| = \|K_n x - K_m x\| \leq \|K_n - K_m\| \, \|x\| \to 0$$

since x is fixed and (1) holds. So $\{y_n\}_n$ is a Cauchy seqeucne in X. By completeness of X we know that there exists $y \in X$ such that

$$y_n \to y.$$

Next we define

$$Kx := y.$$

It is not difficult to prove that $K : X \to X$ and K is linear. So we have

$$\|K_n x - Kx\| \to 0.$$

Which combined with

$$\|K_n x - Kx\| = \lim_{m\to\infty} \|K_n x - K_m x\| \leq \liminf_{m\to\infty} \|K_n - K_m\| \, \|x\|$$

yields $\|K_n - K\| \to 0$, i.e. $K_n \to K$ in $\mathcal{B}(X, X)$.

3. *Proof.* Define

$$K_n = \sum_{i=0}^{n} K^i,$$

for any $n \in \mathbb{N}$. This is well-defined. It is easy to see that

$$\|K^i\| \leq \|K\|^i,$$

whence

$$\begin{aligned} \|K_n - K_m\| &= \|\sum_{i=n+1}^{m} K^i\| \leq \sum_{i=n+1}^{m} \|K\|^i \\ &\leq \|K\|^{n+1} \left(1 + \|K\|^1 + \|K\|^2 + \cdots\right) \\ &\leq \frac{\|K\|^{n+1}}{1 - \|K\|} \to 0. \end{aligned} \tag{2}$$

as $n \to \infty$. Here we assume that $n < m$. Recalling that $\mathcal{B}(X, X)$ we assert that there exists $K \in \mathcal{B}(X, X)$ such that $K_n \to K$ in $\mathcal{B}(X, X)$.

Write

$$(I - K)K_n = \sum_{i=0}^{n} K^i - \sum_{i=1}^{n+1} K^i = I - K^{n+1},$$

so

$$\lim_{n\to\infty} ((I - K)K_n) = (I - K) \lim_{n\to\infty} K_n = I - \lim_{n\to\infty} K^{n+1} = I,$$

since $I - K$ is a bounded operator from X to X. Thus the limit of K_n is just the inverse of $I - K$.

3. *Proof.* Choose a uniformly bounded sequence $\{f_n\}_n$ in $C(\bar{\Omega})$. Invoking the property that a countinuous function on a compact domain is uniformly countinuous, we obtain that $K = K(x, y)$ is uniformly continuous on $\bar{\Omega} \times \bar{\Omega}$. Thus we have for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$|K(x_1, y) - K(x_2, y)| < \varepsilon/M$$

for any $(x_1, y), (x_2, y) \in \bar{\Omega} \times \bar{\Omega}$ satisfying $|x_1 - x_2| < \delta$.

We now define a seuqence $v_n(x) = (Kf_n)(x)$. There hold

i) v_n is uniformly bounded.

ii) v_n is equicontinuous. In fact, we have

$$\begin{aligned} |v_n(x_1) - v_n(x_2)| &= |(Kf_n)(x_1) - (Kf_n)(x_2)| \\ &\leq \int_{\bar{\Omega}} |K(x_1, y) - K(x_2, y)|\,|f_n(y)|dy \\ &\leq \|f_n\|_\infty \text{meas} \cdot \sup |K(x_1, y) - K(x_2, y)| \\ &\leq \varepsilon \end{aligned} \tag{3}$$

here we took $M = \|f_n\|_\infty \text{meas}$.

By Azela-Ascoli K is compact as an operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$.

4. *Proof.* By the definition of compactness, the assertions are not difficult to prove. We thus omit the details here.