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Partial Differential Equations I: Linear Theory

Solutions to the Exercises of Tutorial 11

1. *Proof.* Since $x_0,\ y \in \Gamma$ and Γ is flat, we see that the normal vector n_y, at y, to Γ is orthogonal to Γ. That is

$$(x_0 - y) \cdot n_y = 0. \tag{1}$$

Hereafter the integer upper-scripts as in e.g. y^j, x_0^j, denote the jth component of the vectors y, x_0 respectively. Recalling that for $j = 1, 2, 3$

$$\frac{\partial}{\partial y^j}|x_0 - y| = \frac{y^j - x_0^j}{|x_0 - y|},$$

$$\frac{d}{dr}\frac{e^{i\sqrt{\lambda}r}}{r} = \frac{i\sqrt{\lambda}r - 1}{r^2}e^{i\sqrt{\lambda}r},$$

and that $\frac{\partial}{\partial n_y} = n_y \cdot \nabla_y$, we obtain

$$\begin{aligned} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0 - y|} &= \frac{i\sqrt{\lambda}r - 1}{r^2}e^{i\sqrt{\lambda}r}\bigg|_{r=|x_0-y|} \sum_{i=1}^{3}\left(\frac{\partial}{\partial y^j}|x_0 - y|\right) \cdot n_y^j \\ &= \frac{i\sqrt{\lambda}\,r - 1}{r^2}e^{i\sqrt{\lambda}r}\bigg|_{r=|x_0-y|} \frac{(y - x_0) \cdot n_y}{|x_0 - y|} \\ &= 0, \end{aligned} \tag{2}$$

here, we used (1). And the proof of the assertion is complete.

2. *Solution.* We use the jump relation for the double layer potential

$$\lim_{y \to x} u(y) = -v(x) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0 - y|}v(y)dS_y$$

for $x \in \partial\Omega$.

We define

$$v(y) = \begin{cases} 0, & \text{if } y \in \partial\Omega \setminus \Gamma, \\ -f(y), & \text{if } y \in \Gamma. \end{cases}$$

By the jump relation and (2) we then have that if $x \in \Gamma$,

$$\begin{aligned} \lim_{y \to x} u(y) &= -v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0 - y|}}{|x_0 - y|} v(y) dS_y \\ &= f(x) + \frac{1}{2\pi} \left(\int_{\partial\Omega\setminus\Gamma} + \int_{\Gamma} \right) \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0 - y|}}{|x_0 - y|} v(y) dS_y \qquad (3) \\ &= f(x). \end{aligned}$$

The integrals in (3) is zero since the integrand is equal to zero, due to in first part of the integral $v = 0$ in $\partial\Omega \setminus \Gamma$ while the first factor of the integrand in Γ is equal to zero because of (2). Thus we have constructed a solution to

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \; x \in \Omega, \\ u(x) &= f(x), \; x \in \Gamma. \end{aligned}$$

3. *Solution.* The function u constructed above is, in general, not a solution of the Dirichlet problem

$$\Delta u(x) + \lambda u(x) = 0, \; x \in \Omega, \qquad (4)$$

$$u(x) = f(x), \; x \in \partial\Omega. \qquad (5)$$

The reason is as follows: for $x \in \partial\Omega \setminus \Gamma$, we can not obtain that $u(x) = 0$ which is required by the jump relation. In fact, we have

$$-v(x) + \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x - y|}}{|x - y|} v(y) dS_y = -\frac{1}{2\pi} \int_{\Gamma} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x - y|}}{|x - y|} f(y) dS_y.$$

Since $x \in \partial\Omega \setminus \Gamma$ and $y \in \Gamma$, one can assume reasonably that $\text{dist}\{x, y\} \neq 0$, from which we conclude that $\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}$ is continuous and is not identically equal to zero from the computations in Problem 1. Thus there exists at least one point, say x_0, such that

$$\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x - y|} \neq 0.$$

Thus if we define a continuous function f such that $f(x) = 1$ for all $x \in B_\varepsilon(x_0)$ with ε being suitably small, $f(x) = 0$ for $x \in \mathbb{R}^3 \setminus B_{2\varepsilon}(x_0)$, and $f(x) \in [0, 1]$ for $x \in B_{2\varepsilon} \setminus B_\varepsilon(x_0)$, then there holds for this f that

$$-\frac{1}{2\pi} \int_{\Gamma} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x - y|} f(y) dS_y \neq 0.$$

which by jump relation should be equal to zero since $f(x) = 0$ for $x \in \partial\Omega \setminus \Gamma$. Therefore, u constructed in Problem 3 is not necessary a solution to problem (4) – (5).

4. *Solution.* Since $\Omega = B_R(0) \subset \mathbb{R}^3$, we know that the normal derivative is radial, i.e. $\frac{\partial}{\partial n_y} = \frac{y}{|y|} \cdot \nabla_y$. We consider the real valued solution, whence the jump relation becomes

$$\begin{aligned} \lim_{y \to x} u(y) &= -v(x) + \frac{1}{2\pi} \int_{\partial B_R(0)} \frac{\partial}{\partial n_y} \frac{1}{|x-y|} v(y) dS_y \\ &= -v(x) - \frac{1}{2\pi} \int_{|y|=R} \sum_{j=1}^{3} \frac{y^j}{|y|} \cdot \frac{y^j - x^j}{|x-y|^3} v(y) dS_y \\ &= -v(x) - \frac{1}{2\pi} \int_{|y|=R} \frac{y \cdot (y-x)}{R|x-y|^3} v(y) dS_y. \end{aligned} \tag{6}$$

Suppose that there is a solution u to the boundary value problem

$$\begin{aligned} \Delta u(x) &= 0, \ x \in B_R(0), \\ u(x) &= f(x), \ x \in \partial B_R(0) \end{aligned}$$

where $f = \text{const}$, then f satisfies

$$\text{const} = f(x) = -v(x) - \frac{1}{2\pi} \int_{|y|=R} \frac{y \cdot (y-x)}{R|x-y|^3} v(y) dS_y.$$

We choose that $v(x) = V = \text{const}$, then the above equality becomes

$$\begin{aligned} \text{const} &= -V - \frac{1}{2\pi} \int_{|y|=R} \frac{y \cdot (y-x)}{R|x-y|^3} V dS_y \\ &= -V \left(1 + \frac{1}{2\pi} \int_{|y|=R} \frac{y \cdot (y-x)}{R|x-y|^3} dS_y \right). \end{aligned}$$

from which V can be solved if

$$\frac{1}{2\pi} \int_{|y|=R} \frac{y \cdot (y-x)}{R|x-y|^3} dS_y \neq -1.$$

Thus we prove the existence of solution u.

5. *Solution.* i) Let $z_0 = (0,0,1)$. To compute the integral

$$\frac{1}{2\pi} \int_{|z|=1} \frac{z \cdot (z - z_0)}{|z_0 - z|^3} dS_z,$$

we use the spherical coordinates:

$$\begin{aligned} z_1 &= R \sin\phi \cos\theta, \\ z_2 &= R \sin\phi \sin\theta, \\ z_3 &= R \cos\phi, \end{aligned}$$

where $R = 1$, $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Thus $dS_z = \sin\phi \, d\theta \, d\phi$. Moreover, we have

$$z \cdot (z - z_0) = 1 - \cos\phi, \quad |z_0 - z| = \sqrt{2(1 - \cos\phi)},$$

and the integral turns out to be

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\pi} \frac{1 - \cos\phi}{\left(\sqrt{2(1 - \cos\phi)}\right)^3} \sin\phi \, d\phi &= 2^{-\frac{3}{2}} \int_0^{\pi} \frac{\sin\phi \, d\phi}{\sqrt{1 - \cos\phi}} \\ &= 2^{-\frac{3}{2}} \int_0^2 \frac{dt}{\sqrt{t}} \\ &= 1 \neq -1. \end{aligned} \tag{7}$$

ii) Making a suitable rotation we find that the assumption that $z_0 = (0, 0, 1)$ is not a restriction due to the rotation invariance of this integral. Namely, for any $z_0 \in \partial B_1(0)$, we write it $z_0 = (\sin\phi_0 \cos\theta_0, \sin\phi_0 \sin\theta_0, \cos\phi_0)$, then we make suitable rotation transform M such that

$$M z_0 = (0, 0, 1),$$

However, the form of the integral is not changed.

To do this, we first define

$$M_1 = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 & 0 \\ \sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} \cos\phi_2 & 0 & -\sin\phi_2 \\ 0 & 1 & 0 \\ \sin\phi_2 & 0 & \cos\phi_2 \end{pmatrix},$$

where ϕ_1, ϕ_2 are two constant to be determined. Then define

$$M = M_1 M_2.$$

Letting

$$\phi_1 = -\theta_0, \quad \phi_2 = \phi_0.$$

It is easy to check that $M^T M = I$ and $M z_0 = (0, 0, 1)$. Let $y = Mz$ and $y_0 = M z_0$. There hold $|y| = |Mz| = 1$,

$$z \cdot (z - z_0) = My \cdot (My - z_0) = y \cdot (y - y_0),$$

and $|z - z_0| = |M(y - y_0)| = |y - y_0|$. The integral turns out to be

$$\frac{1}{2\pi} \int_{|y|=1} \frac{y \cdot (y - y_0)}{|y_0 - y|^3} dS_y,$$

which has the exact form as that for z. This allows us to repeat i) for y to evaluate the integral.