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Partial Differential Equations I: Linear Theory
Solutions to the Exercises of Tutorial 10

1. *Proof.* Let $G = G(x, y)$ be the Green function to the Helmholtz equation with $\lambda \in \mathbb{R}$ and the Dirichlet boundary condition, prescribed on the boundary of a bounded open domain $\Omega \subset \mathbb{R}^3$ with smooth boundary.

For any $x, y \in \Omega$ such that $x \neq y$, we define $u(z) = G(z, x),\ v(z) = G(z, y)$. In what follows we freeze temporarily variables x, y and only allow z to change. We see that $u(z), v(z)$ have singularity at $z = x$ or $z = y$, respectively. To remove the singularities, we dig two small balls centered at x, y, i.e. $B_\varepsilon(x),\ B_\varepsilon(y) \subset \Omega$ for a small positive parameter ε. Therefore, $u(z), v(z)$ satisfy the Helmholtz equation in the domain $\Omega_\varepsilon := \Omega \backslash \{B_\varepsilon(x) \cup B_\varepsilon(y)\}$.

Then we can apply the Green formula to $u(z), v(z)$ in Ω_ε and obtain

$$\int_{\Omega_\varepsilon} (v\Delta u - u\Delta v)\, dz \;=\; \int_{\partial\Omega_\varepsilon} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \tag{1}$$

here n_z is the normal vector to Ω_ε.

Since $\Delta_z u + \lambda u = 0$ and $\Delta_z v + \lambda v = 0$ in Ω_ε, we compute the left-hand side as

$$\int_{\Omega_\varepsilon} (v\Delta_z u - u\Delta_z v)\, dz = -\int_{\Omega_\varepsilon} (v \cdot \lambda u - u \cdot \lambda v)\, dz = 0. \tag{2}$$

The boundary of the domain Ω_ε consists of three parts: $\partial\Omega \cup \partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)$, moreover u, v are equal to 0 at the boundary $\partial\Omega$, thus equation (1) turns out to be

$$\begin{aligned} 0 &= \int_{\partial\Omega_\varepsilon} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \\ &= \int_{\partial\Omega \cup \partial B_\varepsilon(x) \cup \partial B_\varepsilon(y)} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \\ &= \int_{\partial B_\varepsilon(x)} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z + \int_{\partial B_\varepsilon(y)} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \\ &=: I_1 + I_2. \end{aligned} \tag{3}$$

Next we want to pass to limit in (3) as $\varepsilon \to 0$, by using the properties (iii) and (iv), listed on Page 68 in the lecture notes, of the Green function G. We first consider I_1. From the definition we see that v is smooth (infinitely continuously differentiable) in $B_\varepsilon(x)$, we write

$$\begin{aligned} I_1 &= \int_{\partial B_\varepsilon(x)} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \\ &= \int_{\partial B_\varepsilon(x)} v(x)\frac{\partial u(z)}{\partial n_z} dS_z + \int_{\partial B_\varepsilon(x)} (v(z) - v(x))\frac{\partial u(z)}{\partial n_z} dS_z \\ &\quad + \int_{\partial B_\varepsilon(x)} u(z)\frac{\partial v}{\partial n_z} dS_z \\ &=: I_{11} + I_{12} + I_{13}. \end{aligned} \tag{4}$$

For I_{11}, using property (iv) on Page 68, we get

$$I_{11} \to v(x). \tag{5}$$

Since v is uniformly continuous on $\partial B_\varepsilon(x)$, for any $\delta > 0$, there exists a small ε, such that

$$|v(z) - v(x)| \le \delta.$$

Thus, I_{12} can be estimated as

$$|I_{12}| \le \delta \int_{\partial B_\varepsilon(x)} \left| \frac{\partial u(z)}{\partial n_z} \right| dS_z \to 0. \tag{6}$$

Here we used that $\int_{\partial B_\varepsilon(x)} \left| \frac{\partial u(z)}{\partial n_z} \right| dS_z$ is finite which can be proved in a similar way as in the lecture notes. (In fact, by definition we can rewrite $u = F + w$ where F is the fundamental solution to the Helmholtz equation and w has good regularity say its derivative in $L_2(\Omega)$, thus $\int_{\partial B_\varepsilon(x)} \left| \frac{\partial w(z)}{\partial n_z} \right| dS_z \to 0$ since $\mathrm{meas}(\partial B_\varepsilon(x)) \to 0$ as $\varepsilon \to 0$. For the term containing F, we can prove that it converges to 0 as in the lecture notes. Since v is smooth, we obtain $|\frac{\partial v}{\partial n_z}| \le C$ on $\partial B_\varepsilon(x)$. Hence

$$|I_{13}| \le C \int_{\partial B_\varepsilon(x)} |u(z)| dS_z \to 0. \tag{7}$$

Here we used $\int_{\partial B_\varepsilon(x)} |u(z)| dS_z \to 0$ which can be proved using a similar idea as above.

Combination of (3) – (7) we then assert that

$$\int_{\partial B_\varepsilon(x)} \left(v\frac{\partial u}{\partial n_z} - u\frac{\partial v}{\partial n_z} \right) dS_z \to v(x) = G(x, y) \tag{8}$$

as $\varepsilon \to 0$.

By symmetric role that (u,x) and (v,y) play, we obtain easily that

$$\int_{\partial B_\varepsilon(y)} \left(v\frac{\partial u}{\partial n_y} - u\frac{\partial v}{\partial n_y} \right) dS_y \to u(y) \tag{9}$$

as $\varepsilon \to 0$.

Therefore, from (3), (8) and (9) we infer that

$$0 = v(x) - u(y),$$

i.e.

$$G(x,y) = G(y,x).$$

And the proof of symmetry of G is complete.

2. *Proof.* We first fix $x \in \Omega$. To guarantee the regularity of G in a certain subdomain of Ω, we also dig a small ball as in Problem 1. Choose a small number $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset \Omega.$$

By definition we see that G satisfies

$$\Delta_y u + \lambda u = 0, \text{ in } \Omega_\varepsilon = \Omega \setminus B_\varepsilon(x),$$

and $u(x,y) = 0$ for $y \in \partial\Omega$.

It is easy to see that $G(x,y) \to +\infty$ as $y \to x$, thus for suitably small ε there holds

$$G(x,y) > 0, \text{ for } y \in \partial B_\varepsilon(x).$$

Recalling that $\lambda < 0$ we can now apply the maximum principle to G over Ω_ε to get

$$G(x,y) \geq \min \left\{ 0, \min_{y \in \partial\Omega_\varepsilon} G(x,y) \right\} = 0,$$

Thus

$$G(x,y) \geq 0, \ y \in \Omega_\varepsilon,$$

since ε is chosen arbitrarily, we also conclude this is true for all $y \in \Omega$ but $y \neq x$. The proof is thus complete.

3. *Proof.* We consider the more general case, i.e. the operator A is defined by

$$A = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right),$$

where a_{ij} are constant and such that $a_{ij} = a_{ij}$ for all $i,j = 1,2,3$.

We want to search a fundamental solution F in the form

$$F = \frac{1}{\kappa} \cdot \frac{1}{|x-y|_M},$$

here κ is a constant to be determined, $|x-y|_M := \sqrt{(x-y)\cdot M(x-y)}$ and $M = (M_{ij})_{3\times 3}$ is a symmetric matrix which should be chosen so that

$$A_x F(x,y) = 0, \quad A_y F(x,y) = 0,$$

and

$$\lim_{r\to 0} \int_{\{y\in\mathbb{R}^3 | \ |x-y| \ = \ r\}} \frac{\partial}{\partial n_y} F(x,y)\, dS_y = 1,$$

where n_y is the unit exterior normal vector.

We now compute $A_x F(x,y)$. To simplify the notations we apply the Einstein summation convention, i.e. repeated indices in a single term are implicitly summed over. In what follows we allow i,j,k,l to take values in $\{1,2,3\}$. Thus, for instance, the operator A can be rewritten as

$$A = a_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$$

and

$$(x-y)\cdot M(x-y) = m_{ij}(x_i - y_i)(x_j - y_j)$$

Recalling $(x^{-\frac{1}{2}})' = -\frac{1}{2}(x^{-\frac{3}{2}}$. Straightforward computations and using the symmetry of M yield

$$\frac{\partial}{\partial x_k} F(x,y) = -\frac{1}{\kappa} \frac{1}{|x-y|_M^3} (m_{kj}(x_j - y_j)) = -\frac{1}{\kappa} \frac{1}{|x-y|_M^3} (m_{ki}(x_i - y_i)),$$

since i,j are dummy indices. And the second order derivatives are

$$\begin{aligned} &\frac{\partial}{\partial x_l} \frac{\partial}{\partial x_k} F(x,y) \\ = \ &\frac{1}{\kappa} \frac{1}{|x-y|_M^5} \Big(3(m_{kj}(x_j - y_j)) \cdot (m_{lj}(x_j - y_j)) - m_{kl} \cdot |x-y|_M^2 \Big). \end{aligned} \tag{10}$$

From the above computations and the definition of A we then obtain

$$\begin{aligned} &A_x F(x,y) \\ = \ &\frac{1}{\kappa} \frac{1}{|x-y|_M^5} \left(3a_{kl}(m_{kj}(x_j - y_j)) \cdot (m_{li}(x_i - y_i) - a_{kl} m_{kl} m_{ij} (x_i - y_i)(x_j - y_j) \right) \\ = \ &\frac{1}{\kappa} \frac{1}{|x-y|_M^5} \left(3(M(x-y)) \cdot \mathcal{A}(M(x-y)) - \mathrm{tr}(\mathcal{A}M)\,(x-y)\cdot M(x-y) \right). \end{aligned} \tag{11}$$

Here, $\mathcal{A} = (a_{ij})$ and the operator tr denotes the trace of a matrix. Therefore, if we choose that

$$M = \mathcal{A}^{-1},$$

then one has

$$\mathrm{tr}(\mathcal{A}M) = \mathrm{tr}(Id) = 3,$$

hence from (11), noting that $M(x-y)\cdot\mathcal{A}M(x-y) = (x-y)\cdot M(x-y)$ by symmetry of M, we conclude that

$$A_x F(x,y) = \frac{3}{\kappa}\frac{1}{|x-y|_M^5}\left(M(x-y)\cdot\mathcal{A}M(x-y) - (x-y)\cdot M(x-y)\right) = 0.$$

Similarly we obtain

$$A_y F(x,y) = 0$$

provided that $M = \mathcal{A}^{-1}$.

It remains to check the third condition in the definition of F. Invoking that n_y to a ball $\{y\in\mathbb{R}^3 \mid |x-y| = r\}$ is radical, we have

$$\begin{aligned}
\frac{\partial}{\partial n_y}F(x,y) &= -\frac{\partial}{\partial r}F(x,y) \\
&= \frac{1}{\kappa}\frac{1}{|x-y|_M^3}(m_{kj}(x_j-y_j))\frac{x_k-y_k}{r} \\
&= \frac{1}{\kappa}\frac{1}{r\,|x-y|_M^3}(x-y)\cdot M(x-y) = \frac{1}{\kappa}\frac{1}{r\,|x-y|_M^3}|x-y|_M^2 \\
&= \frac{1}{\kappa}\frac{1}{r\,|x-y|_M} \qquad (12)
\end{aligned}$$

Thus, transforming the coordinates $y-x$ to (r,ω) with $r = |y-x|$, we obtain

$$\lim_{r\to 0}\int_{\{y\in\mathbb{R}^3\,|\,|x-y|\,=\,r\}}\frac{\partial}{\partial n_y}F(x,y)\,dS_y = \frac{1}{\kappa}\int_{|\omega|=1}\frac{d\omega}{|\omega|_M},$$

from which one gets that the limit is 1 since we choose

$$\kappa = \int_{|\omega|=1}\frac{d\omega}{|\omega|_M}.$$

Therefore, F defined above is a fundamental solution to the operator A.