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AG 06, FB Mathematik
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Partial Differential Equations I: Linear Theory
Solutions to the Exercises of Tutorial 09

1. The picture of Ω_α is easy to draw, we omit it here.
i) To solve the problem

$$\begin{aligned} \Delta u + u &= 0, \text{ in } \Omega_\alpha, \\ u|_{\partial\Omega_\alpha} &= 0, \end{aligned} \tag{1}$$

we first rewrite the equation in the polar coordinates (r, θ), so that we can make use of the method of separation of variables. Recalling

$$\Delta u = \frac{\partial^2}{\partial r^2} u + \frac{1}{r}\frac{\partial}{\partial r} u + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} u,$$

and defining an ansatz

$$u(r, \theta) = R(r)\sin(c\theta),$$

here c is a constant to be determined, we then obtain that a must satisfy

$$\frac{d^2 a}{dr^2} + \frac{1}{r}\frac{da}{dr} + \left(1 - \frac{c^2}{r^2}\right) a = 0, \tag{2}$$

on the other hand, to meet the boundary condition we can choose $c = \frac{n\pi}{2\pi - \alpha}$ so that $\sin(c\theta) = 0$ at the boundary, i.e. $\theta = 0, 2\pi - \alpha$. Since we are intending to find a solution, we just take hereafter

$$c = \frac{\pi}{2\pi - \alpha}.$$

From the lecture notes we know that the solution to (2) can be represented as Bessel's functions, we take one and denote it by $J_c(r)$. Then we obtain a solution u to (1)

$$u(r, \theta) = J_c(r)\sin(c\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(c + k + 1)} \left(\frac{r}{2}\right)^{2k+c} \sin(c\theta).$$

ii) Recalling $c = \frac{\pi}{2\pi-\alpha}$, the fact that the Bessel function converges absolutely and the above formula for u we get easily that

$$u(r,\theta) = r^{\frac{\pi}{2\pi-\alpha}} \sin\left(\frac{\pi}{2\pi-\alpha}\theta\right)\left(\frac{1}{\Gamma(c+1)}\cdot\frac{1}{2^c} + O(r)\right)$$

Since the equation is linear and the boundary condition is homogeneous,

$$\tilde{u}(r,\theta) = \Gamma(c+1)2^c u(r,\theta)$$

is also a solution to the same problem. Thus we find a solution satisfying

$$u(r,\varphi) = r^{\frac{\pi}{2\pi-\alpha}} \sin\left(\frac{\pi}{2\pi-\alpha}\varphi\right)(1+O(r)).$$

2. *Proof.* From the assumption that $f \in L_2(\mathbb{R}^3)$ with $f(x) = 0$ for all $|x| \geq 1$, we infer that $f \in L_1(B_1(0))$ by Cauchy's inequality, thus $\int_{|x|\leq 1} f(x)dx$ exists.

We now turn to prove that for any $|y| \leq 1$ there holds

$$\frac{1}{|x-y|} - \frac{1}{|x|} \to 0,$$

as $|x| \to \infty$. To this end, without loss of generality we may assume $|x| \geq 2$ so that

$$|x-y| \geq |x| - |y| \geq |x| - 1 \geq \frac{1}{2}|x|$$

here we used the triangle inequality and the fact that $|y| \leq 1$. Thus we can rewrite

$$\left|\frac{1}{|x-y|} - \frac{1}{|x|}\right| = \frac{\big|\,|x| - |x-y|\,\big|}{|x-y|\,|x|} \leq \frac{|y|}{|x-y|\,|x|} \leq \frac{1}{\frac{1}{2}|x|^2} \to 0, \tag{3}$$

as $|x| \to \infty$.

Therefore, recalling $u = F * f$ and $F(x) = \frac{1}{4\pi}\frac{1}{|x|}$ (in the real case and in $\mathbb{R}^3$), we obtain

$$\begin{aligned} u(x) &= (F*f)(x) = \frac{1}{4\pi}\int_{|y|\leq 1} \frac{1}{|x-y|} f(y)dy \\ &= \frac{1}{4\pi}\int_{|y|\leq 1}\left(\frac{1}{|x-y|} - \frac{1}{|x|}\right) f(y)dy + \frac{1}{4\pi}\int_{|y|\leq 1}\frac{1}{|x|}f(y)dy. \end{aligned}$$

hence, by (3) and the integrability of f over a bounded domain, one has

$$\begin{aligned} \left||x|u(x) - \frac{1}{4\pi}\int_{|y|\leq 1} f(y)dy\right| &= \left|\frac{1}{4\pi}\int_{|y|\leq 1}\left(\frac{|x|}{|x-y|} - \frac{|x|}{|x|}\right) f(y)dy\right| \\ &\leq \frac{1}{2\pi}\frac{1}{|x|}\int_{|y|\leq 1} |f(y)|dy \\ &\leq \frac{C}{|x|} \to 0, \end{aligned}$$

and the proof is thus complete.

3. *Proof.* Let v be the solution to

$$\Delta v = 0, \text{ in } B_1(0)$$

satisfying the boundary condition $v|_{\partial B_1(0)} = u|_{\partial B_1(0)}$.

Then we define

$$w_\pm = u - v \pm \varepsilon F(r),$$

where $F(r)$ is the non-negative fundamental solution to the Laplace equation and ε is any positive constant. Recalling the property of the non-negative fundamental solution we know that $F(r) \to \infty$ as $x \to 0$. Therefore from the boundedness of u and also v we infer that

$$w_\pm = u - v \pm \varepsilon F(r) \to \pm\infty,$$

as $x \to 0$.

Now we consider a domain $\Omega_\delta = B_1(0) \setminus B_\delta(0)$ with a small constant $\delta << 1$. It is easy to see that $w_\pm$ are harmonic in Ω_δ for any fixed δ. Applying the maximum principle we find

$$\pm w_\pm \geq 0$$

for all $x \in \Omega_\delta$ for arbitrary ε. From which we can conclude that

$$-\varepsilon F(r) \leq u - v \leq \varepsilon F(r),$$

i.e.

$$|u(x) - v(x)| \leq \varepsilon F(r) \to 0,$$

as $\varepsilon \to 0$. Then it follows that

$$u(x) = v(x)$$

for every $x \in \Omega_\delta$. But δ is an arbitrary constant, so $u = v$ in $B_1(0)$. Since v is harmonic, so is u.