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Solutions to the Exercises of Tutorial 08

1. *Proof.* Since u satisfies the mean value formula, we have

$$u(x) = \frac{1}{\pi r^2} \int_{B_r(x)} u(y)dy,$$

for any $B_r(x) \subset \Omega$.

We employ the argument of contradiction and assume that there exists a point $x_0 \in \Omega$, such that $\Delta u(x_0) \neq 0$. Without loss of generality we assume that $\Delta u(x_0) > 0$. From the continuity of u we conclude that there exists a ball $B_\varepsilon(x_0)$ such that

$$\Delta u(x) > 0, \text{ for all } x \in B_\varepsilon(x_0), \tag{1}$$

here ε is a small positive number.

Since we assume that $u \in C_2(\Omega)$, we can apply the Green formula to get

$$\begin{aligned} \int_{B_\varepsilon(x_0)} \Delta u(x)dx &= \int_{\partial B_\varepsilon(x_0)} \nabla u(x) \cdot n dS_x \\ &= \int_0^{2\pi} \frac{\partial}{\partial r} u(r\omega) d\theta \\ &= \frac{\partial}{\partial r} \int_0^{2\pi} u(r\omega) d\theta \\ &= \frac{\partial}{\partial r} u(x_0) = 0. \end{aligned}$$

However, the integral $\int_{B_\varepsilon(x_0)} \Delta u(x)dx$ is positive by (1).

Thus we arrive at a contradiction. Whence $\Delta u(x) = 0$ for all $x \in \Omega$, i.e. u is harmonic.

2. *Proof.* Firstly we compute $\Delta f(u)$ for C^2-functions $f : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}$. There holds

$$\Delta(f(u)) = f'(u)\Delta u + f''(u)|\nabla u|^2.$$

i) Thus choosing $f(u) = u^2$ we then have $\Delta(u^2) = 2\left(u\Delta u + |\nabla u|^2\right)$. Since u is a harmonic function, one has

$$\Delta(u^2) = 2|\nabla u|^2 \geq 0.$$

Now we choose a ball $B \subset\in \mathbb{R}^n$. It is easy to find a solution v to

$$\Delta v = 0, \text{ in } B,$$

and $v|_{\partial B} = (u^2)|_{\partial B}$. We need only to prove that $v \leq u$ in B. To prove this we let $w = u^2$ and $F = w - v$. Then F satisfies

$$\Delta F = g, \text{ in } B,$$

and $F|_{\partial B} = 0$, where $g = 2|\nabla u|^2 \geq 0$. Applying the maximum principle we obtain

$$F \leq 0.$$

Therefore u^2 is sub-harmonic.

ii) For any convex function $f \in C_2(\mathbb{R}, \mathbb{R})$, by a slightly different argument we prove that the composite function $f(u)$ is sub-harmonic.

3. *Proof.* The proof consists of three steps. We first construct a smooth approximation, say $|x|_\varepsilon$, of $|x|$, then prove this approximate function is sub-harmonic. Finally pass the approximate function to limit as $\varepsilon \to 0$.

i) We define

$$|x|_\varepsilon = \sqrt{|x|^2 + \varepsilon^2}.$$

It is easy to show that $|x| + \sqrt{|x|^2 + \varepsilon^2} \geq \varepsilon$ for all x. Thus

$$|\ |x|_\varepsilon - |x|\ | = \frac{\varepsilon^2}{|x|_\varepsilon + |x|} \leq \varepsilon \to 0$$

as $\varepsilon \to 0$.

ii) Straightforward computations yield

$$\partial_{x_i}(|x|_\varepsilon) = \frac{x_i}{|x|_\varepsilon},$$

and

$$\partial^2_{x_i}(|x|_\varepsilon) = \frac{|x|^2 + \varepsilon^2 - x_i^2}{|x|^3_\varepsilon},$$

from which we obtain

$$\Delta(|x|_\varepsilon) = \frac{(n-1)|x|^2 + n\varepsilon^2}{|x|^3_\varepsilon} =: f_\varepsilon(x).$$

Choose any ball $B \subset \mathbb{R}^n$. Suppose that v^ε is a solution to

$$\Delta v^\varepsilon = 0, \text{ in } B,$$

and $v^\varepsilon|_{\partial B} = (|x|_\varepsilon)|_{\partial B}$. From the lecture notes we know such solution exists. Let $w_\varepsilon = |x|_\varepsilon - v^\varepsilon$, then w satisfies

$$\Delta w = f, \text{ in } B,$$

and $w_\varepsilon|_{\partial B} = 0$. Note that $f_\varepsilon \geq 0$. Applying the maximum principle yields

$$w \leq 0$$

whence $|x|_\varepsilon$ is sub-harmonic.

iii) $|x|_\varepsilon$ converges to $|x|$ uniformly. Note that

$$|x| \leq |x|_\varepsilon \leq v^\varepsilon, \text{ in } B$$

and $(v^\varepsilon)|_{\partial B} = |x|_\varepsilon^\varepsilon|_{\partial B} \to (|x|)|_{\partial B}$ as $\varepsilon \to 0$. We also have $v^\varepsilon \to v$ in $C(\bar{B})$ which is harmonic. Thus $|x|$ is sub-harmonic. This argument indicates that Theorem 5.12 in the lecture notes is still true for the case that we have infinitely many sub-solutions.

4. i) Define $w \equiv 0$. It is easy to see that w is a super-solution to the equation

$$\lambda v + \Delta v = 0, \quad \text{in } \Omega,$$

with $\lambda \leq 0$ satisfying the boundary condition $w|_{\partial\Omega} = 0$.

ii) There are two cases according to $\lambda = 0$ or $\lambda < 0$.

a) We assume firstly that $\lambda < 0$. Define a function in the form:

$$u_1(x) = -M\left(1 - e^{-\mu d(x)}\right)$$

Here M, $\mu > 0$ are positive constants to be determined later on, and $d = d(x)$ defined $d(x) = \text{dist}(x, \partial\Omega)$ for any $x \in \Omega$, is the distance function which is Lipschitz continuous (this can be proven by using the triangle inequality easily). Furthermore, since the boundary $\partial\Omega$ is C^2, we have that $d(x)$ is C^2 too for x near $\partial\Omega$, u_1 is also C^2. Now we calculate

$$\partial_i u_1 = -\mu M e^{-\mu d}\partial_i d,$$

and

$$\partial_i^2 u_1 = \left(\mu(\partial_i d)^2 - \partial_i^2 d\right) M\mu e^{-\mu d},$$

Therefore, we obtain for x near $\partial\Omega$, say $x \in \Omega_\gamma = \left\{x \in \Omega \mid d(x) < \frac{1}{\gamma}\right\}$, that

$$\begin{aligned}
\lambda u_1 + \Delta u_1 &= -\lambda M + \left(\lambda + \mu^2\sum_{i=1}^{3}(\partial_i d)^2 - \mu\Delta d\right) M e^{-\mu d} \\
&\geq -\lambda M + \frac{1}{2}\lambda M \\
&= -\frac{1}{2}\lambda M > 0,
\end{aligned}$$

provided that $x \in \Omega_\gamma$ and $\gamma = \frac{1}{\mu}$, where μ is sufficiently small from which it follows that $e^{-\mu d}$ approaches to 1 (In fact, we have $e^{-1} \leq e^{-\mu d} \leq 1$ for all $x \in \Omega_\gamma$), also the continuity of Δd and $\partial_i d$ over $\bar{\Omega}_\gamma$ which implies that they are uniformly bounded on $\bar{\Omega}_\gamma$, namely there exists a constant K such that $|\partial_i d| + |\Delta d| \leq K$.

The above computations motivate us to construct a sub-solution as follows: Choose the two positive constants M, C such that $C << M$. Then we have

$$\underline{u}(x) = u_1(x)$$

for x which closes sufficiently to the boundary of Ω, otherwise

$$\underline{u}(x) = -C.$$

Using the property that $\max\{f, g\}$ is a sub-solution if f, g are sub-solutions (see also Theorem 5.12 in the lecture notes), one can easily prove that $\underline{u}$ is a sub-solution. to the Helmholtz equation, satisfying $\underline{u}|_{\partial\Omega} = 0$. It remains to show that u_1 is a sub-solution.

Since d may be not differentiable at x which is not near the boundary, we employ an approximate procedure. One way is to apply the Weierstrass theorem which says that a continuous function can be approximated by a polynomial sequence. Another is to use the convolution introduced in the lecture. Here we use the latter one. Taking a function $\varphi_\varepsilon \in C_\infty$ such that $\int_{\mathbb{R}^n} \varphi_\varepsilon dx = 1$, the support of $\varphi_\varepsilon(x)$ is $B_\varepsilon(0)$. It is not difficult to prove that

$$f * \varphi_\varepsilon \in C_\infty(\mathbb{R}^n),$$

if $f \in L_{1,\text{loc}}(\mathbb{R}^n)$. Applying this to function d which is Lipschitz continuous, hence $d \in L_1(\Omega)$, so $d * \varphi_\varepsilon \in C_\infty(\Omega)$. Moreover, we have

$$d * \varphi_\varepsilon \to d, \text{ in } C(\Omega).$$

We now define a new function u_1^ε by

$$u_1^\varepsilon(x) = -M\left(1 - e^{-\mu d^\varepsilon(x)}\right)$$

Similar the computations carried out for u_1, and choosing μ suitably small (however maybe depends on ε in this case) yield

$$\begin{aligned} \lambda u_1^\varepsilon + \Delta u_1^\varepsilon &= -\lambda M + \left(\lambda + \mu^2 \sum_{i=1}^{3} (\partial_i d^\varepsilon)^2 - \mu \Delta d^\varepsilon\right) M e^{-\mu d^\varepsilon} \\ &\geq -\lambda M + \frac{1}{2}\lambda M \\ &= -\frac{1}{2}\lambda M > 0, \end{aligned}$$

for any fixed ε. From this one can conclude that u_1^ε is a sub-solution to Helmholtz equation satisfying suitable boundary condition. Using again Theorem 5.12 in a slightly general version (i.e. for infinitely many sub-solutions), we then conclude the limit, u_1, of u_1^ε, is also a sub-solution to the Helmholtz equation satisfying the zero boundary condition.

b) We now consider the case that $\lambda = 0$. For this problem, the above technique does not work. However it is valid for a new function v^ε which is defined by $v^\varepsilon = e^{\varepsilon x_1} u$ with a small parameter ε. For simplicity we omit the upper-script ε till we investigate its limit. Then one has

$$\partial_1 v = \varepsilon e^{\varepsilon x_1} u + e^{\varepsilon x_1} \partial_1 u, \quad \partial_i v = e^{\varepsilon x_1} \partial_i u, \text{ for } i \neq 1,$$

and

$$\partial_1^2 v = \varepsilon^2 e^{\varepsilon x_1} u + 2\varepsilon e^{\varepsilon x_1} \partial_1 u + e^{\varepsilon x_1} \partial_1^2 u, \quad \partial_i^2 v = e^{\varepsilon x_1} \partial_i^2 u, \text{ for } i \neq 1,$$

Thus

$$\Delta v + 2\varepsilon \partial_1 v - 3\varepsilon^2 v = 0.$$

There is a term in the equation, which involves the first order derivative $\partial_1 v$. However, from the argument in a) we see that this is not an essential difficulty. Letting $\lambda = -3\varepsilon^2$, we see that this equation satisfies the requirement of a), i.e. $\lambda < 0$.

Therefore we can construct a sub-solution satisfying $v^\varepsilon|_{\partial\Omega} = 0$, in a similar manner as in a), for every fixed ε. Then letting $\varepsilon \to 0$ we get the limit function u satisfying $u|_{\partial\Omega} = 0$ and u is a sub-solution to the Helmholtz equation.