Prof. Dr. H. D. Alber

WS 07/08
30. 11. 07

AG 06, FB Mathematik
Tech. Univ. Darmstadt

Solutions to the Exercises of Tutorial 07

1. *Proof.* Using polar coordinates and the mean value formula we rewrite

$$\begin{aligned}\int_{|x-y|\le R} u(y)dy &= \int_0^R \int_0^{2\pi} u(r\omega)\, dS_y dr \\ &= \int_0^R 2\pi r u(x)\, dr \\ &= \pi u(x) R^2.\end{aligned}$$

Here ω is a unit vector defined by $\frac{x}{|x|}$ and dS_y is the surface element.
Thus we obtain

$$u(x) = \frac{1}{\pi R^2} \int_{|x-y|\le R} u(y)\, dy.$$

2. *Proof.* a) Applying the Green formula we have

$$\begin{aligned}0 &= \int_{B_r(x)} \Delta u dx = \int_{B_r(x)} \Delta u \cdot 1\, dx \\ &= -\int_{B_r(x)} \nabla u \cdot \nabla 1 dx + \int_{\partial B_r(x)} \nabla u \cdot n\, dS_y \\ &= \int_{\partial B_r(x)} \nabla u \cdot n\, dS_y.\end{aligned}$$

Noting that the normal vector n is radical, thus we obtain $\nabla u \cdot n = \frac{\partial}{\partial r}$, and we have

$$0 = \int_0^{2\pi} \frac{\partial}{\partial r} u(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta. \tag{1}$$

b) We now integrate the equation in a) with respect to r over $(0, R)$ to get

$$\begin{aligned}
0 &= \int_0^R \int_0^{2\pi} \frac{\partial}{\partial r} u(x_1 + r\cos\theta, x_2 + r\sin\theta)\, d\theta dr \\
&= \int_0^{2\pi} \int_0^R \frac{\partial}{\partial r} u(x_1 + r\cos\theta, x_2 + r\sin\theta)\, dr d\theta \\
&= \int_0^{2\pi} u(x_1 + r\cos\theta, x_2 + r\sin\theta)|_0^R \, d\theta \\
&= \int_0^{2\pi} u(x_1 + R\cos\theta, x_2 + R\sin\theta) d\theta - 2\pi u(x_1, x_2).
\end{aligned}$$

From which we obtain the mean value formula.

3. *Proof.* Let $x_1, x_2 \in \mathbb{R}^2$ be two different points. Without loss of generality, by choosing a suitable coordiante system we may assume that x_1, x_2 are located at the x-axis, i.e. $x_1 = (x^1, 0)$, $x_2 = (x^2, 0)$. Furthermore we assume that $x^1 < x^2$. In what follows, we denote $y = (y^1, y^2)$.

By the mean value formula, we write

$$u(x_1) - u(x_2) = \frac{1}{\pi r^2} \left(\int_{|x_1 - y| \le r} u(y) dy - \int_{|x_2 - y| \le r} u(y) dy \right), \tag{2}$$

here the right-hand side can be rewritten as

$$\frac{1}{\pi r^2} \left(\int_{B_r(x_1) \backslash B_r(x_2)} u(y) dy - \int_{B_r(x_2) \backslash B_r(x_1)} u(y) dy \right).$$

Next we want to calculate the two quantities which appeared in the above equality, i.e. $\text{meas}(B_r(x_2) \backslash B_r(x_1))$ and $\text{meas}(B_r(x_1) \backslash B_r(x_2))$, we need only, by symmetry, compute one of them. We have

$$\text{meas}(B_r(x_2) \backslash B_r(x_1)) = \text{meas}(B_r(x_2)) - \text{meas}(B_r(x_2) \cap B_r(x_1)).$$

Since it is easy to see $\text{meas}(B_r(x_2)) = \pi r^2$, it remains to calculate $\text{meas}(B_r(x_2) \cap B_r(x_1))$. Note that

$$\begin{aligned}
B_r(x_2) \cap B_r(x_1) &= \{ y \in \mathbb{R}^2 \mid |x_1 - y| \le r,\ |x_2 - y| \le r \} \\
&= \{ y \in \mathbb{R}^2 \mid (x^1 - y^1)^2 + (y^2)^2 \le r^2,\ (x^2 - y^1)^2 + (y^2)^2 \le r^2 \},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
B_r(x_2) \cap B_r(x_1) = \Big\{ y \in \mathbb{R}^2 \mid & x^1 - \sqrt{r^2 - (y^2)^2} \le y^1 \le x^1 + \sqrt{r^2 - (y^2)^2}, \\
& x^2 - \sqrt{r^2 - (y^2)^2} \le y^1 \le x^2 + \sqrt{r^2 - (y^2)^2} \Big\}.
\end{aligned}$$

From the assumption that $x^1 < x^2$, we then arrive at

$$B_r(x_2) \cap B_r(x_1) = \left\{ y \in \mathbb{R}^2 \mid x^2 - \sqrt{r^2 - (y^2)^2} \le y^1 \le x^1 + \sqrt{r^2 - (y^2)^2} \right\}.$$

From this, it therefore, follows that

$$
\begin{aligned}
\operatorname{meas}(B_r(x_2)\cap B_r(x_1)) &= \int_{B_r(x_2)\cap B_r(x_1)} 1\,dy \\
&= \int_{\left\{y\in\mathbb{R}^2 | x^2-\sqrt{r^2-(y^2)^2}\le y^1\le x^1+\sqrt{r^2-(y^2)^2}\right\}} 1\,dy \\
&= \int_{-r}^{r} dy_2 \int_{x^2-\sqrt{r^2-(y^2)^2}}^{x^1+\sqrt{r^2-(y^2)^2}} 1\,dy^1 \\
&= \int_{-r}^{r} \left((x^1-x^2)+2\sqrt{r^2-(y^2)^2}\right)\,dy^2 \\
&= \pi r^2 - 2(x^2-x^1)r. \qquad (3)
\end{aligned}
$$

Here we observed that the integral $\int_{-r}^{r} 2\sqrt{r^2-(y^2)^2}dy^2$ is just the area of a disk $B_r(0)\subset\mathbb{R}^2$, hence it is equal to πr^2.

Now we recall (2) and the assumption that u is bounded in $\mathbb{R}^2$ (thus there holds $M=\sup_{x\in\mathbb{R}^2}|u(x)|<\infty$) to get

$$
\begin{aligned}
u(x_1)-u(x_2) &= \frac{1}{\pi r^2}\left(\int_{B_r(x_1)\setminus B_r(x_2)} u(y)dy - \int_{B_r(x_2)\setminus B_r(x_1)} u(y)dy\right) \\
&\le \frac{M}{\pi r^2}\Big(\operatorname{meas}(B_r(x_1)\setminus B_r(x_2)) + \operatorname{meas}(B_r(x_2)\setminus B_r(x_1))\Big) \\
&\le \frac{M}{\pi r^2}\cdot 4r(x^2-x^1) \le \frac{C}{r} \to 0,
\end{aligned}
$$

as $r\to\infty$. Here C is a positive constant which is independent of r.

Thus we conclude for any two points x_1, x_2 that there holds $u(x_1)=u(x_2)$. Hence u is a constant function.