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AG 06, FB Mathematik
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Solutions to the Exercises of Tutorial 06

1. *Solution.* It is easy to see that

$$u(x) = \sin(\pi x)$$

is a solution to the problem. From $\sin(\pi/2) = 1$ we find that $u(x)$ achieves its maximum 1 at $x = \frac{1}{2} \in (0, 1)$. Suppose that the maximum principle were true, noting that in this case we have $f = 0$ in $(0, 1)$ and

$$\max_{x\in\partial\Omega} u(x) = \min_{x\in\partial\Omega} u(x) = 0,$$

we then have

$$u(x) \le \max\left(0, \max_{x\in\partial\Omega} u(x)\right) = 0,$$

and

$$u(x) \ge \min\left(0, \min_{x\in\partial\Omega} u(x)\right) = 0,$$

thus $u = 0$ in $[0, 1]$ which is a contradiction.

If we employ the strong maximum principle, then $u(x)$ achieves a positive maximum 1 at an interior point $x = \frac{1}{2}$, which contradicts $u(x) \le 0$ or $u(x) < \max_{x\in\partial\Omega} u(x) = 0$.

Therefore, the maximum principle fails for this equation.

2. *Proof.* We define

$$g(x) = u^2(x),$$

then the equation can be rewritten as

$$\Delta u - g\, u = f.$$

By a similar argument as in the proof of the maximum principle for the linear equation, we can conclude easily that

$$u(x) \le \max\left(0, \max_{y\in\partial\Omega} u(y)\right), \text{ if } f(x) \ge 0 \text{ in } \Omega,$$

$$u(x) \geq \min\left(0, \min_{y\in\partial\Omega} u(y)\right), \text{ if } f(x) \leq 0 \text{ in } \Omega.$$

We now prove the uniqueness of solution to the Dirichlet boundary value problem for this nonlinear equation. Define

$$w(x) = u(x) - v(x),$$

one derives that w satisfies

$$\Delta w - g\,w = 0, \text{ in } \Omega$$

and the boundary condition

$$w = 0, \text{ on } \partial\Omega,$$

where $g = u^2 + uv + v^2$. Note that $f = 0$. It is easy to prove that g is non-negative, from which we know that the conditions for the maximum principle are met. Therefore,

$$\max_{y\in\partial\Omega} w(y) = \min_{y\in\partial\Omega} w(y) = 0,$$

thus

$$w(x) \leq \max(0, 0) = 0,$$

and

$$w(x) \geq \min(0, 0) = 0,$$

so

$$w(x) = 0$$

for all $x \in \Omega$, that is $u \equiv v$ on $\bar{\Omega}$.

3. *Proof.* i) The proof is similar to that in the lecture by replacing the function $|x|^2$ by e^{Nx_1}, where $N > 0$ is a sufficiently large number. However we proceed here in a different way. The key insight is that we can prove easily the maximum principle in the case that $f > 0$ and we can construct a suitable perturbation of u, say v, so that the right-hand side of the equation of v is strictly positive, and the difference between the maxima of v and u converges to 0 as ε goes to zero.

Step 1. Assume that the right hand side f is strictly positive in Ω.

Suppose that there exists $x_0 \in \Omega$ such that $u(x_0) > 0$ and

$$\max_{x\in\bar{\Omega}} u(x) = u(x_0).$$

At x_0, there hold

$$\frac{\partial u}{\partial x_i}(x_0) = 0, \quad \frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0, \quad i = 1, \cdots, n,$$

hence,

$$0 < f(x_0) = \Delta u(x_0) + \sum_{k=1}^{n} a_i(x_0)\frac{\partial}{\partial x_i}u(x_0) - g(x_0)u(x_0) = \Delta u(x_0) - g(x_0)u(x_0) \leq 0.$$

This is a contradiction. Thus u can not achieve its positive maximum at a point in Ω.

Step 2. Now we apply the maximum principle in the version in Step 1 to a small perturbation of u, say v. Define

$$v(x) = u(x) + \varepsilon w(x)$$

where w is a function to be determined. Straightforward computations yield

$$\Delta v = \Delta u + \varepsilon \Delta w, \quad \frac{\partial}{\partial x_i} v = \frac{\partial}{\partial x_i} u + \varepsilon \frac{\partial}{\partial x_i} w, \; i = 1, \cdots, n,$$

whence by using the equation of u we obtain

$$\begin{aligned} & \Delta v + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} v - gv \\ = \; & \Delta u + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} u - gu + \varepsilon \left(\Delta w + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} w - gw \right) \\ = \; & f + \varepsilon \left(\Delta w + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} w - gw \right) \\ = \; & \tilde{f}. \end{aligned} \tag{1}$$

Now we make suitable choice of the function w. By continuity on $\bar{\Omega}$ (Here we assume that $\bar{\Omega}$ is a bounded set) of g, a_i for $i = 1, \cdots, n$, we can assume that there exists a positive constant M such that

$$|g(x)|, \quad |a_i(x)| \leq M, \; i = 1, \cdots, n$$

for all $x \in \bar{\Omega}$. Defining

$$w(x) = e^{N x_1},$$

here N is a constant such that $N^2 > (N+1)M$ which is true for a sufficiently large N . Thus

$$\begin{aligned} \Delta w + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} w - gw &= \left(N^2 + a_1 N - g\right) w \\ &\geq \left(N^2 - MN - M\right) w \\ &= \left(N^2 - (N+1)M\right) w \\ &> 0 \end{aligned}$$

which yields

$$\tilde{f}(x) > 0$$

for all $x \in \bar{\Omega}$. From (1) applying the assertion obtained in Step 1 to v we then obtain

$$v(x) \leq \max \left(0, \max_{y \in \partial\Omega} v(y) \right),$$

from which by definition we infer that

$$u(x) + \varepsilon w(x) \leq \max \left(0, \max_{y \in \partial\Omega} \{u(y) + \varepsilon w(y)\} \right),$$

this holds for any positive ε. Therefore, recalling that w is independent of ε and taking limit as $\varepsilon \to 0$ one has

$$u(x) \leq \max \left(0, \max_{y \in \partial\Omega} u(y) \right).$$

Step 3. For the other assertion, Making a transformation $w = -u$, applying the result in Step 2 to w, we obtain the conclusion for the case that $f \leq 0$.

ii) For the second part, the proof is the same as in the lecture. So we omit the details.