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AG 06, FB Mathematik
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Solutions to the Exercises of Tutorial 05

1. *Proof.* a) By definition we have

$$z^m \mathcal{J}_m(z) = \sum_{k=0}^{\infty} \frac{2^m(-1)^k}{k!(m+k)!}\left(\frac{z}{2}\right)^{2k+2m},$$

thus recalling that $\mathcal{J}_m(z)$ is entire, we obtain

$$\begin{aligned}
\frac{d}{dz}\{z^m \mathcal{J}_m(z)\} &= \sum_{k=0}^{\infty} \frac{2^m(-1)^k(2k+2m)}{k!(m+k)!}\left(\frac{z}{2}\right)^{2k+2m-1}\cdot\frac{1}{2} \\
&= \sum_{k=0}^{\infty} \frac{2^m(-1)^k}{k!(m+k-1)!}\left(\frac{z}{2}\right)^{2k+2m-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k-1)!}\left(\frac{z}{2}\right)^{2k+m-1}\left(\left(\frac{z}{2}\right)^m\cdot 2^m\right) \\
&= z^m \mathcal{J}_{m-1}(z),
\end{aligned}$$

so we get $\frac{d}{dz}\{z^m \mathcal{J}_m(z)\} = z^m \mathcal{J}_{m-1}(z)$.

b) Again from the definition of the Bessel function we infer that

$$\begin{aligned}
&\mathcal{J}_{m+1}(z) + \mathcal{J}_{m-1}(z) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+1+k)!}\left(\frac{z}{2}\right)^{2k+m+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m-1+k)!}\left(\frac{z}{2}\right)^{2k+m-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+1+k)!}\left(\frac{z}{2}\right)^{2k+m+1} \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(m-1+k)!}\left(\frac{z}{2}\right)^{2k+m-1} + \frac{1}{(m-1)!}\left(\frac{z}{2}\right)^{m-1}.
\end{aligned}$$

We then make a transformation $k-1=j$ i.e. $k=j+1$ to get

$$
\begin{aligned}
& \mathcal{J}_{m+1}(z)+\mathcal{J}_{m-1}(z) \\
= & \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+1+k)!}\left(\frac{z}{2}\right)^{2k+m+1} \\
& +\sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(j+1)!(m+j)!}\left(\frac{z}{2}\right)^{2j+m+1}+\frac{1}{(m-1)!}\left(\frac{z}{2}\right)^{m-1} \\
= & \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+1+k)!}\left(\frac{z}{2}\right)^{2k+m+1} \\
& +\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!(m+k)!}\left(\frac{z}{2}\right)^{2k+m+1}+\frac{1}{(m-1)!}\left(\frac{z}{2}\right)^{m-1} \\
= & \sum_{k=0}^{\infty}\left(\frac{1}{m+1+k}-\frac{1}{k+1}\right) \frac{(-1)^k}{(k+1)!(m+k)!}\left(\frac{z}{2}\right)^{2k+m+1}+\frac{1}{(m-1)!}\left(\frac{z}{2}\right)^{m-1} \\
= & \sum_{k=0}^{\infty} \frac{-m}{(k+1)(m+1+k)} \frac{(-1)^k}{(k+1)!(m+k)!}\left(\frac{z}{2}\right)^{2k+m+1}+\frac{1}{(m-1)!}\left(\frac{z}{2}\right)^{m-1} \\
= & \sum_{k=0}^{\infty} \frac{m(-1)^{k+1}}{(k+1)!(m+k)!}\left(\frac{z}{2}\right)^{2k+m+1}+\frac{m}{m!}\left(\frac{z}{2}\right)^{m-1} \\
& (\text{changing } k+1=j) \\
= & \sum_{j=1}^{\infty} \frac{m(-1)^j}{j!(m+j-1)!}\left(\frac{z}{2}\right)^{2j+m-1}+\frac{m}{m!}\left(\frac{z}{2}\right)^{m-1} \\
= & \frac{2m}{z} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(m+j-1)!}\left(\frac{z}{2}\right)^{2j+m}=\frac{2m}{z} \mathcal{J}_m(z).
\end{aligned}
$$

To prove the third assertion of b), we employ assertion a). Computation yields

$$
\frac{d}{dz}\{z^m \mathcal{J}_m(z)\}=m z^{m-1} \mathcal{J}_m(z)+\frac{d}{dz} \mathcal{J}_m(z) z^m=z^m \mathcal{J}_{m-1}(z),
$$

dividing both sides of the above equality one has

$$
z \mathcal{J}_m^{\prime}(z)+m \mathcal{J}_m(z)=\mathcal{J}_{m-1}(z).
$$

This is the third assertion.

From the first and third assertions we obtain the second conclusion, and from the first and second assertions, the fourth follows.

c) We are now going to prove the last assertion. Similar to the lecture, we introduce

$$
u_{m,i}(r)=\mathcal{J}_m(k_{m,i} r), \text{ and } u_{m,j}(r)=\mathcal{J}_m(k_{m,j} r),
$$

Then $u_{m,l}(r)$ (here $l = i, j$) satisfies

$$u''_{m,l}(r) + \frac{1}{r}u'_{m,l}(r) + \left((k_{m,l})^2 - \frac{m^2}{r^2}\right)u_{m,l}(r) = 0. \tag{1}$$

We define

$$Au(r) = -\left(u''(r) + \frac{1}{r}u'(r) - \frac{m^2}{r^2}u(r)\right),$$

it is easy to see that A is not self-adjoint operator, and we can regard $(k_{m,l})^2$ as an eigenvalue of it, since equation (1) can be rewritten as $Au_{m,l}(r) = (k_{m,l})^2 u_{m,l}(r)$.

We now multiply (1) (choosing $l = i$) by $ru_{m,j}(r)$, integrate the resulting equation with respect to r over $(0, 1)$ and use twice integration by parts to obtain

$$\begin{aligned}
0 &= \int_0^1 ru''_{m,i}u_{m,j} + u'_{m,i}u_{m,j} + \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) ru_{m,i}u_{m,j}dr \\
&= \int_0^1 (ru'_{m,i})'u_{m,j} + \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) u_{m,i}u_{m,j}dr \\
&= ru'_{m,i}u_{m,j}|_0^1 - \int_0^1 ru'_{m,i}u'_{m,j} + \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) ru_{m,i}u_{m,j}dr \\
&= -ru_{m,i}u'_{m,j}|_0^1 + \int_0^1 u_{m,i}(ru'_{m,j})' + \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) ru_{m,i}u_{m,j}dr \\
&= \int_0^1 u_{m,i}(ru'_{m,j})' + \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) ru_{m,i}u_{m,j}dr \\
&= -\int_0^1 \left((k_{m,j})^2 - \frac{m^2}{r^2}\right) ru_{m,j}u_{m,i}dr \\
&\quad + \int_0^1 \left((k_{m,i})^2 - \frac{m^2}{r^2}\right) ru_{m,i}u_{m,j}dr.
\end{aligned}$$

Here we used the fact that $k_{m,l}$ $(l = i, j)$ are zeros of $\mathcal{J}_m$ from which it follows that the bounary terms vanish. We also used the equation for $u_{m,j}(r)$.

Whence we can obtain

$$\left((k_{m,j})^2 - (k_{m,i})^2\right) \int_0^1 ru_{m,i}u_{m,j}dr = 0,$$

which implies $\int_0^1 ru_{m,i}(r)u_{m,j}(r)dr = 0$ provided $(k_{m,j})^2 \neq (k_{m,i})^2$. And the proof of c) is complete.

2. *Proof.* i) Using the fact that

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}, \text{ for } |q| < 1,$$

and choosing $q = \frac{re^{i(\varphi-\theta)}}{R}$, we then have $|q| = \frac{r}{R} < 1$, thus

$$\begin{aligned} u(x) &= u(r,\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi} \\ &= \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} u^{(b)}(\theta) \left(\frac{r}{R}\right)^{|m|} e^{im(\varphi-\theta)} d\theta \\ &\quad \text{(by the absolute convergence and } a_m = 0,\ m < 0) \\ &= \frac{1}{2\pi} \int_0^{2\pi} u^{(b)}(\theta) \sum_{m=0}^{\infty} \left(\frac{r}{R}\right)^{|m|} e^{im(\varphi-\theta)} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u^{(b)}(\theta) \frac{1}{1-\frac{re^{i(\varphi-\theta)}}{R}} d\theta \\ &= \frac{R}{2\pi} \int_0^{2\pi} \frac{u^{(b)}(\theta)}{R - re^{i(\varphi-\theta)}} d\theta. \end{aligned}$$

So we prove the first part.

ii) Note that the assumption $a_m = 0$ for $m < 0$ can be met provided $u^{(b)}$ is analytic. We regard $f(Re^{i\theta})$ in the Cauchy integral formula as $u^{(b)}(\theta)$, write $w = Re^{i\theta}$ and $z = re^{\varphi}$, then from the definition of Γ we infer that

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z} dw &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\theta})}{Re^{i\theta} - re^{i\varphi}} iRe^{i\theta} d\theta \\ &= \frac{R}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{R - re^{i(\varphi-\theta)}} d\theta \\ &= f(z). \end{aligned}$$

Here we used i). It is easy to infer i) from the Cauchy integral formula. Q.E.D.

3. *Proof.* It is easy to see that g(t) is infinite times continuously differentiable for any t which satisfies $t > 0$ or $t < 0$. We thus need only investigate the differentiability, at $t = 0$, of $g(t)$. To this end, we invoke a theorem: The exponent function $\exp(x)$ grows, as $x \to +\infty$, faster than any polynomial say $P_n(x)$, i.e.

$$\frac{|P_n(x)|}{\exp(x)} \to 0, \text{ as } x \to \infty. \tag{2}$$

To prove this, we recall the expansion

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq \frac{x^m}{m!}$$

for any $x \geq 0$. Here $m \in \mathbb{N}_0$ is arbitrary. Suppose the degree of $P_n(x)$ is n and the coefficient of the principal term is a_n, we then choose $m > n$ and arrive at

$$\frac{|P_n(x)|}{\exp(x)} \leq \frac{(2a_n x^n + C)m!}{x^m} = \frac{2a_n m!}{x^{m-n}} + \frac{Cm!}{x^m} \to 0 \tag{3}$$

Step 1. We prove that $g'(t)$ is continuous. We first calculate the derivative of it for $t > 0$ and $t < 0$. It is easy to get $g'(t) = 0$ if $t < 0$, while $g'(t) = \frac{C}{t^2} e^{-\frac{1}{t}} = (t^{-1})^2 g(t)$ if $t > 0$. Thus we can find a polynomial $P_1(t) = t^2$ and write $g'(t) = P_1(t^{-1})g(t)$ if $t > 0$. By using (2) we arrive at easily $g'(t) \to 0$ as $t \to 0^+$, thus

$$g'(0^-) = 0 = g'(0^+).$$

Step 2. Suppose that $g^{(n)}(t)$ is continuous, and there exists a polynomial $P_n(t^{-1})$ such that

$$g^n(t) = \begin{cases} P_n(t^{-1})g(t), & \text{if } t > 0, \\ 0, & \text{if} t < 0. \end{cases} \tag{4}$$

Step 3. Similar to Step 1 we can prove that $g^{(n+1)}(t)$ is continuous.

Therefore, $g(t)$ is infinite times continuous differentiable since $g^n(t)$ is continuous for any $n \in \mathbb{N}$.