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Solutions to the Exercises of Tutorial 04

1. *Proof.* We employ the mathematical induction to prove the Leibniz formula.

Step 1. Suppose that $\alpha = 0$. Then the left hand side is equal to $D^\alpha(u\varphi) = u\varphi$ while β in the right hand side is equal to 0 since there hold $\alpha = 0$ and $\beta \le \alpha$, thus $\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta u \cdot D^{\alpha-\beta}\varphi = u\varphi$. Therefore, the Leibniz formula is true for $\alpha = 0$.

Step 2. Let α satisfy $|\alpha| \le k-1$ where $k \in \mathbb{N}$. Suppose the Leibniz formula is true for this α.

Step 3. Now let $\alpha = \gamma + \delta$ where $|\gamma| \le k-1$ and $|\delta| = 1$, it is easy to see that δ can take values in $\{\delta \in \mathbb{N}_0^n \mid \sum_{j=1}^n \delta^j = 1\}$ (This means that only one of δ^i is equal to 1 and the others are equal to 0). Thus, applying step 2 we have

$$\begin{aligned} D^\alpha(u\varphi) &= D^\gamma D^\delta(u\varphi) = D^\gamma \sum_{i=1}^n D_{x_i}(u\varphi) \\ &= D^\gamma \sum_{i=1}^n D_{x_i}u\varphi + D^\gamma \sum_{i=1}^n uD_{x_i}\varphi \\ &= \sum_{i=1}^n \sum_{\beta\le\gamma}\binom{\gamma}{\beta}D^\beta D_{x_i}u \cdot D^{\gamma-\beta}\varphi + \sum_{i=1}^n \sum_{\beta\le\gamma}\binom{\gamma}{\beta}D^\beta u \cdot D^{\gamma-\beta}D_{x_i}\varphi \quad (1) \\ &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta u \cdot D^{\alpha-\beta}\varphi. \end{aligned}$$

Here, we used the formula that

$$\binom{\gamma}{\beta-\delta} + \binom{\gamma}{\beta} = \binom{\gamma+\delta}{\beta} = \binom{\alpha}{\beta}$$

when we calculated the coefficient of the term $D^{\gamma+\delta}uD^\beta\phi$ which comes from the term $D^\gamma D_{x_i}uD^\beta\phi$ with coefficient $\binom{\gamma}{\beta-\delta}$, and the term $D^\gamma uD^\beta D_{x_i}\phi$ with coefficient $\binom{\gamma}{\beta}$ in equality (1).

2. *Proof.* a) To prove $uv \in H_1((a,b))$ for any given $u, v \in H_1((a,b))$, we need only prove that

$$\|uv\|_{L_2((a,b))} \le C \text{ and } \|D_x(uv)\|_{L_2((a,b))} \le C$$

provided that $\|u\|_{H_1((a,b))} \le C$ and $\|u\|_{H_1((a,b))} \le C$, where we use the notation $\|\cdot\|_{H_1((a,b))}$ for the norm of $H_1((a,b))$ which is defined by $\|f\|_{H_1((a,b))} = (\|f\|^2_{L^2((a,b))} + \|D_x f\|^2_{L^2((a,b))})^{\frac{1}{2}}$. Making use the inequality listed in the hint of this exercise, we obtain

$$\int_a^b |uv|^2 dx \le \left(\sup_{x\in(a,b)} |u(x)| \right)^2 \int_a^b |v|^2 dx \le C.$$

For the second assertion we use the Leibniz formula and conclude that $D_x(uv) = (D_x u)\, v + u\, (D_x v)$, thus

$$\begin{aligned} \int_a^b |D_x(uv)|^2 dx &\le 2 \left(\int_a^b |(D_x u) v|^2 dx + \int_a^b |(D_x v) u|^2 dx \right) \\ &\le 2 \left(\sup_{x\in(a,b)} |v(x)| \right)^2 \int_a^b |D_x u|^2 dx + 2 \left(\sup_{x\in(a,b)} |u(x)| \right)^2 \int_a^b |D_x v|^2 dx \\ &\le C. \end{aligned}$$

Here, we used the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$. So $H_1((a,b))$ is a Banach algebra.

b) i) Choose a function $u \in C_\beta(\Omega)$. We are going to prove that $u \in C_\alpha(\Omega)$ for any $\beta \ge \alpha$. To this end, we write

$$\begin{aligned} \|u\|_{C_\alpha(\Omega)} &= \|u\|_{L_\infty(\Omega)} + \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \\ &= \|u\|_{L_\infty(\Omega)} + \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\beta} \cdot |x-y|^{\beta-\alpha} \\ &\le \|u\|_{L_\infty(\Omega)} + C \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|u(x)-u(y)|}{|x-y|^\beta} \\ &\le C\|u\|_{C_\beta(\Omega)}, \end{aligned}$$

where C is a constant depending on the diameter of Ω. From this we assert $\|u\|_{C_\alpha(\Omega)} \le C$ which implies $u \in C_\alpha(\Omega)$.

ii) By assumption $u \in C_\alpha(\Omega)$ and $v \in C_\beta(\Omega)$, we then obtain

$$\begin{aligned} \|uv\|_{C_\gamma(\Omega)} &= \|uv\|_{L_\infty(\Omega)} + \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|u(x)v(x)-u(y)v(y)|}{|x-y|^\gamma} \\ &\le \|u\|_{L_\infty(\Omega)}\, \|v\|_{L_\infty(\Omega)} + \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|u(x)-u(y)|\, |v(x)|}{|x-y|^\alpha} |x-y|^{\alpha-\gamma} \\ &\quad + \sup_{x\in\Omega} \sup_{x\ne y,\ y\in\Omega} \frac{|v(x)-v(y)|\, |u(y)|}{|x-y|^\beta} |x-y|^{\beta-\gamma} \\ &\le \|u\|_{L_\infty(\Omega)}\, \|v\|_{L_\infty(\Omega)} + C([u]_\alpha + [v]_\beta) \\ &\le \|u\|_{C_\alpha(\Omega)} \|v\|_{C_\beta(\Omega)}, \end{aligned}$$

here, we have used that $|x-y|^{\beta-\gamma}$, $|x-y|^{\alpha-\gamma} \le C$ since $\alpha-\gamma \ge 0$ and $\beta-\gamma \ge 0$ by assumption $\gamma = \min\{\alpha, \beta\}$.

iii) We calculate

$$
\begin{aligned}
[F(v)]_\gamma(\Omega) &= \sup_{x\in\Omega}\sup_{x\neq y,\ y\in\Omega}\frac{|F(v(x))-F(v(y))|}{|x-y|^\gamma}\\
&= \sup_{x\in\Omega}\sup_{v(x)\neq v(y),\ \text{for } x\neq y,\ y\in\Omega}\frac{|F(v(x))-F(v(y))|}{|x-y|^\gamma}\\
&= \sup_{x\in\Omega}\sup_{x\neq y,\ y\in\Omega}\frac{|F(v(x))-F(v(y))|}{|v(x)-v(y)|^\alpha}\frac{|v(x)-v(y)|^\alpha}{|x-y|^\gamma}\\
&\le \sup_{x\in\Omega}\sup_{v(x)\neq v(y),\ y\in\Omega}\frac{|F(v(x))-F(v(y))|}{|v(x)-v(y)|^\alpha}\sup_{x\in\Omega}\sup_{x\neq y,\ y\in\Omega}\left(\frac{|v(x)-v(y)|}{|x-y|^\beta}\right)^\alpha\\
&\le C.
\end{aligned}
$$

Then it is easy to conclude that $\|F(v)\|_{C_\gamma(\Omega)} \le C$. Q.E.D.

3. *Solution.* We say a function $u \in H_1(\Omega)$ is a *weak solution* to the Neumann boundary value problem if for any $v \in H_1(\Omega)$ (or $\varphi \in C_\infty(\Omega)$), the following equality holds

$$-(\nabla_x u, \nabla_x v) + \lambda(u, v) = (f, v). \tag{2}$$

i) Suppose that $u \in C_2^*(\Omega) \cap C_1(\bar\Omega)$ is a classical solution to the Neumann boundary value problem (NBVP), we will prove that u satisfies (2). Multiplying the Helmholtz equation by $v \in H_1(\Omega)$, integrating by parts and making use of the boundary condition we arrive at

$$
\begin{aligned}
(f, v) &= \lambda(u, v) + (\Delta u, v)\\
&= \lambda(u, v) - (\nabla_x u, \nabla_x v) + \int_{\partial\Omega}\frac{\partial u}{\partial n} v dS\\
&= \lambda(u, v) - (\nabla_x u, \nabla_x v).
\end{aligned}
$$

Therefore u is also a weak solution to the (NBVP).

ii) We now assume that $u \in H_1(\Omega)$ is a *weak solution* to the (NBVP) and $u \in C_2^*(\Omega) \cap C_1(\bar\Omega)$, we will show that u is classical. That means we need to prove that the equation and the boundary conditions are satisfied. We first prove the first assertion. Choose any test function $v \in \overset{0}{H_1}(\Omega)$ since $\overset{0}{C_\infty}(\Omega) \subset H_1(\Omega)$. In this case v vanishes at the boundary. Then from (2) by integration by parts we infer that

$$(f, v) = \lambda(u, v) + (\Delta u, v) - \int_{\partial\Omega}\frac{\partial u}{\partial n} v dS = \lambda(u, v) + (\Delta u, v),$$

which is

$$(\Delta u + \lambda u - f, v) = 0.$$

This is still true for all $v \in \overset{0}{H_1}(\Omega)$ because $\overset{0}{C_\infty}(\Omega)$ is dense in $\overset{0}{H_1}(\Omega)$. By the fundamental lemma of calculus of variation we obtain

$$\Delta u + \lambda u - f = 0,$$

so the equation is satisfied.

Now we choose test functions in $H_1(\Omega)$, integrate by parts from (2) one then has

$$(\Delta u, v) - \int_{\partial\Omega} \frac{\partial u}{\partial n} v dS + \lambda(u, v) = (f, v). \tag{3}$$

Since we have proved that the equation is satisfied, equation (3) is reduced to

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} v dS = 0,$$

we restrict v to the boundary, then $v|_{\partial\Omega}$ is an arbitrary continuous function defined on $\partial\Omega$. Thus by the assertion of exercise 1 in Tutorial 1, we conclude that $\frac{\partial u}{\partial n} = 0$, so u is a classical solution to the Neumann boundary value problem. Q.E.D.

4. *Proof.* We write

$$|u|^q = |u|^{q\theta} \cdot |u|^{q(1-\theta)},$$

and choose d, d' such that $d = \frac{p}{q\theta}$ and $d' = \frac{r}{q(1-\theta)}$, so

$$\frac{1}{d} + \frac{1}{d'} = 1,$$

then apply the Hölder inequality to get

$$\begin{aligned} \int_\Omega |u|^q &= \int_\Omega |u|^{q\theta} |u|^{q(1-\theta)} \\ &\le \left(\int_\Omega |u|^{q\theta d}\right)^{\frac{1}{d}} \left(\int_\Omega |u|^{q(1-\theta)d'}\right)^{\frac{1}{d'}} \\ &= \left(\int_\Omega |u|^p\right)^{\frac{1}{d}} \left(\int_\Omega |u|^r\right)^{\frac{1}{d'}} \end{aligned} \tag{4}$$

By the definition of $L_p(\Omega)-$norm i.e. $\|f\|_{L_p(\Omega)} = (\int_\Omega |f|^p dx)^{\frac{1}{p}}$ for $p \ge 1$, taking the $q-$th roots of the both sides of inequality (4), noting that the function $x \mapsto x^{\frac{1}{q}}$ (with $x \ge 0$ for $p > 0$), we then arrive at the inequality

$$\|u\|_{L_q(\Omega)} \le \|u\|^{\theta}_{L_p(\Omega)} \|u\|^{1-\theta}_{L_r(\Omega)}.$$