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AG 06, FB Mathematik
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Solutions to the Exercises of Tutorial 03

1. *Solution.* Suppose that u is a solution to the eigenvalue problem. Integrating the equation over Ω and making use of the boundary condition yields

$$u'|_a^b + \lambda \int_\Omega u dx = 0.$$

Thus we have

$$\int_\Omega u\, dx = 0$$

provided that $\lambda \neq 0$. This is the necessary condition for the existence of solution to this problem with $\lambda \neq 0$.

To solve this problem, we consider first $\lambda = 0$, for which we have the general solutions

$$u(x) = C_1 x + C_2.$$

So $u'(x) = C_1$. By the boundary conditions we assert that $C_1 = 0$. Therefore, $u(x) = C_2$.

If $\lambda \neq 0$, the general solutions become

$$u(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x},$$

and

$$u'(x) = C_1\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - C_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$$

By the boundary conditions we thus obtain

$$\begin{aligned} C_1 e^{\sqrt{-\lambda}a} - C_2 e^{-\sqrt{-\lambda}a} &= 0, \\ C_1 e^{\sqrt{-\lambda}b} - C_2 e^{-\sqrt{-\lambda}b} &= 0. \end{aligned}$$

To guarantee non-trivial solutions to the above linear system of (C_1, C_2), the determinant of its coefficient matrix must vanish, which requires

$$1 - e^{2\sqrt{-\lambda}(a-b)} = 0,$$

so the eigenvalues are

$$\lambda_m = \left(\frac{\pi m}{b-a}\right)^2, \text{ for } m \in \mathbb{N}.$$

and the corresponding eigenfunctions are

$$u_m(x) = \cos\left(\frac{\pi m}{b-a}(x-a)\right).$$

(A straightforward computation shows the necessary condition is met.) Similar to the argument in the lecture, it is easy to show $\{u_m\}_{m=0}^{\infty}$ is a complete basis in $L^2(\Omega)$.

2. *Proof.* Roughly speaking, the *regularity* of solutions to a well-posed problem of a PDE means the solutions become more regular provided that the given data (such as initial data, boundary data, or the given function(s) in the equation) is more regular.

Assume that u is a classical solution to the boundary value problem in excercise 2, from this we only know that $u \in C_2(\Omega) \cap C(\bar{\Omega})$ and u satisfies the equation and the boundary conditions. No any information is known on the derivatives of order which is greater than 2. However we can obtain higher regularity of this solution from the equation and the assumption that $f \in C_m(\bar{\Omega})$ by the *bootstrap argument* and the *difference quotient* technique.

Step 1. Let $m = 0$. So we have $f,\, u \in C_0(\bar{\Omega})$. From the equation we can obtain easily that $u'' \in C_0(\bar{\Omega})$ which implies also that $u' \in C_0(\bar{\Omega})$ since we have $u'(x) = u'(a) + \int_a^x u''(y)dy$. Whence,

$$u \in C_2(\bar{\Omega}).$$

Step 2. Let $m = 1$, this means $f \in C_1(\bar{\Omega})$. We define for a continuous function $g : \Omega \to \mathbb{R}$ the difference quotient

$$g_h(x) = \frac{g(x+h) - g(x)}{h}, \text{ for } h \neq 0.$$

Choose a point $x \in \Omega$ (The case that $x \in \partial\Omega$ is slightly different, the argument is left to the reader.) Let h satisfy that $h < \frac{1}{2}\operatorname{dist}\{x, \partial\Omega\}$ so that $x + h \in \Omega$. Then

$$\begin{aligned} u''(x+h) + \lambda u(x+h) &= f(x+h), \\ u''(x) + \lambda u(x) &= f(x), \end{aligned}$$

taking the difference of the above two equations and dividing the resulting equation by h we get

$$(u'')_x(x) + \lambda u_h(x) = f_h(x),$$

The first term in this equation is well-defined since u'' is continuous in Ω. We infer from the assumptions that

$$(u'')_h(x) = f_h(x) - \lambda u_h(x) \to f'(x) - \lambda u'(x),$$

as $h \to 0$. Thus $u''' + \lambda u' = f'$ in Ω, and $u''' \in C(\Omega)$. Moreover, from the equation we can conclude $u''' \in C(\bar{\Omega})$.
Step 3. Suppose the conclusion is true for $m = k-1$ with $k \in \mathbb{N}$. Define

$$v = D^{k-1}u, \ g = D^{k-1}f.$$

Repeating the procedure in Step 2 for v and g, we prove easily the assertion for arbitrary $m \in \mathbb{N}$.

3. *Proof.* Let $\varphi \in \overset{\circ}{C}{}^{\infty}(\Omega)$ be a test function. We have

$$(D_x(cf), \varphi) = -(cf, D_x\varphi) = -c(f, D_x\varphi) = c(D_xf, \varphi) = (cD_xf, \varphi),$$

so $D_x(cf) = cD_xf$. We write

$$\begin{aligned}(D_x(f+g), \varphi) &= -(f+g, D_x\varphi) = -(f, D_x\varphi) - (g, D_x\varphi)\\ &= (D_xf, \varphi) + (D_xg, \varphi) = (D_xf + D_xg, \varphi),\end{aligned}$$

thus the second assertion is proved. Since $\varphi \in \overset{\circ}{C}{}^{\infty}(\Omega)$ one has

$$D_xD_y\varphi = D_yD_x\varphi.$$

Thus

$$(D_xD_yu, \varphi) = (u, D_yD_x\varphi) = (u, D_xD_y\varphi) = (D_yD_xu, \varphi),$$

from which we prove $D_xD_yu = D_yD_xu$.

4. *Proof.* Note that $X = \mathrm{span}\{f_\alpha\}_{\alpha \in E}$ is a sub-space of L^2, using the projection theorem we conclude that there exists a point $Pf \in X$ such that

$$(f - Pf, x) = 0, \ \text{for all } x \in X. \tag{1}$$

Thus

$$\begin{aligned}\|f - x\|^2 &= \|(f - Pf) + (Pf - x)\|^2\\ &= \|f - Pf\|^2 + (f - Pf, Pf - x) + (Pf - x, f - Pf) + \|Pf - x\|^2\\ &= \|f - Pf\|^2 + \|Pf - x\|^2\\ &\geq \|f - Pf\|^2,\end{aligned}$$

the equality holds only if $x = Pf$. Suppose that $Pf = \sum_{\alpha \in E} u_\alpha f_\alpha$, inserting this into (1) and replacing x by f_α with $\alpha \in E$, yield

$$0 = (f - \sum_{\alpha \in E} u_\alpha f_\alpha, f_\alpha) = (f, f_\alpha) - u_\alpha(f_\alpha, f_\alpha) = (f, f_\alpha) - u_\alpha,$$

whence $u_\alpha = (f, f_\alpha) = c_\alpha$ for $\alpha \in E$.

5. *Solution.* We first prove the uniqueness. If there are two pairs (y_1^1, y_2^1) and (y_1^2, y_2^2), such that

$$x = y_1^1 + y_2^1, \ \text{and} \ x = y_1^2 + y_2^2$$

Thus $(y_1^1 - y_1^2) + (y_2^1 - y_2^2) = 0$. So

$$(y_1^1 - y_1^2, y_1^1 - y_1^2) = -(y_2^1 - y_2^2, y_1^1 - y_1^2) = 0,$$

which implies that $\|y_1^1 - y_1^2\|^2 = 0$, thus

$$y_1^1 = y_1^2, \text{ also } y_2^1 = y_2^2.$$

And the uniqueness follows.

Next we prove the existence of such a decomposition for any $x \in X$. There are two cases that should be taken into account.

i) Assume that $x \in Y$. Just write $x = y_1 + 0$, done.

ii) The case that $x \notin Y$. By the projection theorem, we assert that there exists a point $Px \in Y$ such that

$$(x - Px, y) = 0, \text{ for all } y \in Y.$$

We now write $x = (x - Px) + Px$, clearly, $x - Px \in Y^\perp$ and $Px \in Y$, thus we get the representation, i.e. $x = y_1 + y_2$ with $y_1 \in Y^\perp$, $y_2 \in Y$.

Therefore, form the above argument we have $X = Y + Y^\perp$.

Remark. The operator P is a projection operator upon Y, such that $P^2 = P$.