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AG 06, FB Mathematik
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Solutions to the Exercises of Tutorial 02

1. *Solution.* We write a multi-index α_i with $i \in \mathbb{N}_0$, in the following form

$$\alpha_i = (\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^n).$$

a) For the second order operator, we have

$$\sum_{|\alpha|\leq 2} c_\alpha(x) D^\alpha u(x) = \sum_{i=0}^{2} \sum_{\sum_{j=0}^n \alpha_i^j = i} c_{\alpha_i^1, \alpha_i^2, \cdots, \alpha_i^n}(x) D^{\alpha_i} u(x).$$

For the third order operator, one gets easily that

$$\sum_{|\alpha|=3} c_\alpha(x) D^\alpha u(x) = \sum_{\sum_{j=0}^n \alpha_i^j = 3} c_{\alpha_3^1, \alpha_3^2, \cdots, \alpha_3^n}(x) D^{\alpha_3} u(x).$$

b) Let $(x_1, x_2) = (x, t)$. Define $\alpha_2 = (2,0),\ (0,0),\ (0,0),\ (0,2)$, and $c_{\alpha_2}(x) = -c^2,\ 0,\ 0,\ 1$ correspondingly. Then

$$u_{tt} - c^2 u_{xx} = \sum_{|\alpha|=2} D^\alpha u(x,t).$$

Similarly, for the Laplace equation, we let $x = (x_1, x_2)$, then define $\alpha_2 = (2,0)$, $(0,0),\ (0,0),\ (0,2)$, and $c_{\alpha_2}(x) = 1,\ 0,\ 0,\ 1$ correspondingly. Then we have

$$\Delta u = \sum_{|\alpha|=2} D^\alpha u.$$

2. *Proof.* From the formula of u we calculate

$$u_t(x,t) = w(x,t;\tau)|_{t=\tau} + \int_0^t w_t(x,t;\tau)d\tau = \int_0^t w_t(x,t;\tau)d\tau, \qquad (1)$$

here, we used the first initial data. Thus

$$u_{tt}(x,t) = w_t(x,t;\tau)|_{t=\tau} + \int_0^t w_{tt}(x,t;\tau)d\tau = f(x,t) + \int_0^t w_{tt}(x,t;\tau)d\tau,$$

where we made use of the second initial data. It is easy to show that

$$u_{xx}(x,t) = \int_0^t w_{xx}(x,t;\tau)d\tau.$$

Therefore we obtain

$$\begin{aligned} u_{tt}(x,t) - c^2 u_{xx}(x,t) &= f(x,t) + \int_0^t w_{tt}(x,t;\tau)d\tau - c^2 \int_0^t w_{xx}(x,t;\tau)d\tau \\ &= f(x,t) + \int_0^t \left(w_{tt}(x,t;\tau) - c^2 w_{xx}(x,t;\tau)\right) d\tau \\ &= f(x,t). \end{aligned}$$

So the equation is satisfied. Next we should check that the initial and boundary conditions are satisfied. It is easy to prove that the boundary conditions are met. For the first initial condition for u, we invoke the formula of u, then get $u(x,t)|_{t=0} = \int_0^t w(x,t;\tau)d\tau|_{t=0} = 0$,. From (1) we obtain $u_t(x,t)|_{t=0} = \int_0^t w_t(x,t;\tau)d\tau|_{t=0} = 0$. So the initial conditions are satisfied.

3. *Solution.* By the projection theorem, we can conclude that for an element $z \in X \setminus Y$, there exists a point $z_0 \in Y$ such that

$$g(y) := (z - z_0, y) = 0, \text{ for all } y \in Y.$$

Thus $(z - z_0, z_0) = 0$, and $(z - z_0, z) = (z - z_0, z - z_0) = \|z - z_0\|^2$.

Now we define

$$f(x) = \frac{(z - z_0, x)}{(z - z_0, z - z_0)},$$

it is easy to see that $f(z) = 1$ and $f(y) = 0$ for all $y \in Y$. Finally, we show that f is a bounded functional. By the Cauchy inequality, we have

$$|f(x)| \le \frac{\|z - z_0\| \, \|x\|}{\|z - z_0\|^2} \le C\|x\|.$$

Here C is a positive constant depending on z. Q.E.D.

4. *Proof.* a) *Approach 1.* By the convexity of the function $x \mapsto \exp(x)$, we have

$$\begin{aligned} ab &= \exp(\log a + \log b) = \exp\left(\frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)\right) \\ &\le \frac{1}{p}\exp(\log(a^p)) + \frac{1}{q}\exp(\log(b^q)) \\ &= \frac{1}{p}a^p + \frac{1}{q}b^q. \end{aligned}$$

From which the Young inequality follows.

Approach 2. i) Let's first consider the case that $f(a) = b$, i.e. $a^{p-1} = b$. It is easy to see that

$$\begin{aligned} \frac{1}{p}a^p + \frac{1}{q}b^q &= \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy \\ &= \int_0^a x^{p-1}dx + \int_0^{a^{p-1}} y^{q-1}dy. \end{aligned}$$

However making a transformation of variables $y^{q-1} = x$, noting that $(p-1)(q-1) = 1$ or $p - 1 = \frac{1}{q-1}$, we then get

$$y = x^{p-1}, \text{ thus } dy = (p-1)x^{p-2}dx,$$

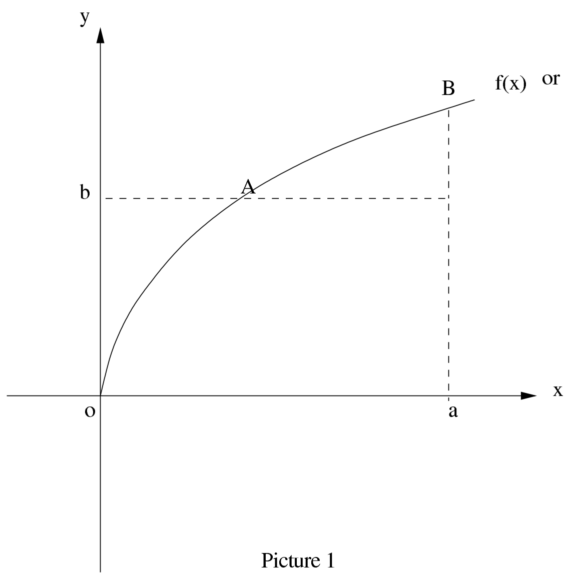
hence,

$$\int_0^{a^{p-1}} y^{q-1}dy = \int_0^a (p-1)x^{p-2} \cdot xdx = \int_0^a (p-1)x^{p-1}dx.$$

Therefore,

$$\frac{1}{p}a^p + \frac{1}{q}b^q = \int_0^a x^{p-1}dx + \int_0^a (p-1)x^{p-1}dx = \int_0^a px^{p-1}dx = a^p = ab.$$

ii) Suppose that i) does not happen, then we have $f(a) > b$ or $f(a) < b$. Since $f(a) < b$ is equivalent to $g(b) > a$, and the functions f, g or the two numbers a, b play a symmetric role in the Young inequality, we need only to deal with the case that $f(a) > b$, i.e. $a > b^{q-1}$. See Picture 1 below,



Picture 1

We write

$$\begin{aligned}\frac{1}{p}a^p + \frac{1}{q}b^q &= \int_0^a x^{p-1}dx + \int_0^b y^{q-1}dy \\ &= \int_{b^{q-1}}^a x^{p-1}dx + \int_0^{b^{q-1}} x^{p-1}dx + \int_0^b y^{q-1}dy \\ &= I_1 + I_2 + \int_0^b y^{q-1}dy. \qquad (2)\end{aligned}$$

It can be see easily from Picture 1 that $\int_0^a x^{p-1}dx$ is the area of the region $oABao$ and $\int_0^b y^{q-1}dy$ the area of the region $obAo$.

We now estimate I_1, I_2. From the fact that x^{p-1} is monotone over $\mathbb{R}^+$, we infer that

$$I_1 \geq \int_{b^{q-1}}^a (b^{q-1})^{p-1}dx = \int_{b^{q-1}}^a bdx = b(a - b^{q-1}) = ab - b^q.$$

For I_2, we handle it in a similar way as in i) and obtain

$$I_2 = \int_0^b (q-1)y^{q-1}dy.$$

Combination the above two estimates with (2) yields

$$\begin{aligned}\frac{1}{p}a^p + \frac{1}{q}b^q &\geq ab - b^q + \int_0^b (q-1)y^{q-1}dy + \int_0^b y^{q-1}dy \\ &= ab - b^q + \int_0^b qy^{q-1}dy = ab - b^q + y^q|_0^b \\ &= ab,\end{aligned}$$

which implies the Young inequality.

b) Applying the Young inequality to

$$a = \frac{f}{\|f\|_{L^p}}, \ b = \frac{g}{\|g\|_{L^q}},$$

(here we used the notation $\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}$.) then we have

$$\left|\frac{f}{\|f\|_{L^p}} \frac{g}{\|g\|_{L^q}}\right| = ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q = \frac{1}{p}\frac{|f|^p}{\|f\|_{L^p}^q} + \frac{1}{q}\frac{|g|^q}{\|g\|_{L^q}^q},$$

so the function $\frac{f}{\|f\|_{L^p}} \frac{g}{\|g\|_{L^q}}$ is integrable. Integrating the above equation with respect to x over Ω, one can obtain easily that

$$\int_\Omega \left|\frac{f}{\|f\|_{L^p}} \frac{g}{\|g\|_{L^q}}\right| dx \leq \frac{1}{p}\int_\Omega \frac{|f|^p}{\|f\|_{L^p}^q}dx + \frac{1}{q}\int_\Omega \frac{|g|^q}{\|g\|_{L^q}^q}dx = \frac{1}{p} + \frac{1}{q} = 1.$$

From this the Hölder inequality follows.