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Solutions to the Exercises of Tutorial 01

1. a) *Proof.* We employ the argument of contradiction. Assume that there exists at least one point, say $x_0 \in [a, b]$, such that

$$f(x_0) \neq 0. \tag{1}$$

Without loss of generality, we may assume that

$$f(x_0) > 0.$$

By continuity of f, we assert that there exists a small positive number ε which satisfies

$$f(x) > 0, \text{ for all } x \in (x_0 - \varepsilon, x_0 + \varepsilon).$$

Thus

$$f(x) \geq \min_{x \in \left[x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\right]} f(x) > 0, \text{ for all } x \in \left[x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\right].$$

Next we are going to construct a suitable test function φ that vanishes at the boundary $x = a, b$. We consider the case that $x_0 \in (a, b)$. When $x_0 = a$ or b, we can treat it in a slightly different way.

$$\varphi(x) = \begin{cases} 0, & \text{for } \ a \leq x < x_0 - \varepsilon, \\ \frac{f(x_0)}{\varepsilon}(x - (x_0 - \varepsilon)) & \text{for } \ x_0 - \varepsilon \leq x < x_0, \\ -\frac{f(x_0)}{\varepsilon}(x - (x_0 + \varepsilon)) & \text{for } \ x_0 \leq x < x_0 + \varepsilon, \\ 0, & \text{for } \ x_0 + \varepsilon \leq x \leq b. \end{cases}$$

(see Pic. 1) Therefore one obtains that

$$\varphi(x) \geq \frac{f(x_0)}{2}, \quad \text{for } x \in \left[x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}\right],$$

whence

$$\begin{aligned}
\int_a^b f(x)\varphi(x)dx &= \int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\varphi(x)dx \geq \int_{x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} f(x)\varphi(x)dx \\
&\geq \int_{x_0-\frac{\varepsilon}{2}}^{x_0+\frac{\varepsilon}{2}} \left(\min_{x\in[x_0-\frac{\varepsilon}{2},x_0+\frac{\varepsilon}{2}]} f(x)\right)\cdot \frac{f(x_0)}{2}dx \\
&= \varepsilon \cdot \frac{f(x_0)}{2} \cdot \min_{x\in[x_0-\frac{\varepsilon}{2},x_0+\frac{\varepsilon}{2}]} f(x) \\
&\geq 0, \qquad (2)
\end{aligned}$$

this contradicts to $\int_a^b f(x)\varphi(x)dx = 0$. So assumption (1) is wrong. Q. E. D.

b) *Solution.* In this case, we can only conclude that

$$f(x) = 0, \text{ for all } x \in \Omega,$$

however it is allowed that f differs from zero on the boundary.

2. *Proof.* We first check that the initial data are satisfied. Let $\xi = x + ct$, and $\eta = x - ct$. From the d'Alembert formula, one can obtain easily

$$u(x,0) = \frac{1}{2}\left(u_0(x) + u_0(x)\right) + \frac{1}{2c}\int_x^x u_1(\xi)d\xi = u_0(x),$$

and

$$\begin{aligned}
u_t(x,0) &= \frac{1}{2}\left(u_0'(\xi)\frac{\partial\xi}{\partial t}|_{t=0} + u_0'(\eta)\frac{\partial\xi}{\partial t}|_{t=0}\right) + \frac{1}{2c}\left(u_1(\xi)\frac{\partial\xi}{\partial t} - u_1'(\eta)\frac{\partial\eta}{\partial t}\right)|_{t=0} \\
&= \frac{1}{2}(cu_0'(x) - cu_0'(x)) + \frac{1}{2c}\left(cu_1(x) + cu_1(x)\right) \\
&= u_1(x). \qquad (3)
\end{aligned}$$

Straightforward computations yield

$$\begin{aligned}
u_{tt} &= \frac{c^2}{2}\left(u_0''(x+ct) + u_0''(x-ct)\right) + \frac{c}{2}\left(u_1'(x+ct) - u_1'(x-ct)\right) \\
u_{xx} &= \frac{1}{2}\left(u_0''(x+ct) + u_0''(x-ct)\right) + \frac{1}{2c}\left(u_1'(x+ct) - u_1'(x-ct)\right),
\end{aligned}$$

from which it follows that

$$u_{tt} - c^2 u_{xx} = 0.$$

Thus the wave equation is satisfied.

How to get the **d'Alembert** *formula?* One approach is to make use of the ansatz $u(x,t) = f(x+ct) + g(x-ct)$ and the initial data.

$$u_0(x) = u(x,0) = f(x) + g(x), \quad u_1(x) = u_t(x,0) = c(f'(x) - g'(x)). \qquad (4)$$

Next we solve f, g in terms of u_0, u_1. To this end, we integrate the second equation in (4) to get

$$f(x) - g(x) = \frac{1}{c}\int^x u_1(\xi)d\xi + C,$$

here C is a constant. Combination with the first equation in (4) yields

$$f(x) = \frac{1}{2}\left(u_0(x) + \frac{1}{c}\int^x u_1(\xi)d\xi + C\right),\ g(x) = \frac{1}{2}\left(u_0(x) - \frac{1}{c}\int^x u_1(\xi)d\xi - C\right).$$

Then from the ansatz we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(u_0(x+ct) + \frac{1}{c}\int^{x+ct} u_1(\xi)d\xi + C\right) \\ &\quad + \frac{1}{2}\left(u_0(x-ct) - \frac{1}{c}\int^{x-ct} u_1(\xi)d\xi - C\right) \\ &= \frac{1}{2}\left(u_0(x+ct) + u_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_1(\xi)d\xi, \end{aligned} \tag{5}$$

which is the d'Alembert formula.
3. *Proof.* i) Since we assume that $u = u(x,t)$ is a twice continuously differentiable function, there holds

$$u_{xt} = u_{tx}.$$

By definition we have

$$\begin{aligned} D_+D_-u &= (D_-u)_t + c(D_-u)_x = (u_t - cu_x)_t + c(u_t - cu_x)_x \\ &= u_{tt} - cu_{xt} + cu_{tx} - c^2u_{xx} \\ &= u_{tt} - c^2u_{xx}. \end{aligned}$$

Thus

$$\Box u = D_+D_-u.$$

In a same manner one can easily prove

$$\Box u = D_-D_+u.$$

ii) Define $v = D_-u$, then we infer from $\Box u = 0$ and i) that $D_+v = 0$. On the other hand, from $v = D_-u,\ D_+v = 0$ one has $\Box u = 0$. That is $\Box u = 0$ is equivalent to $v = D_-u,\ D_+v = 0$.