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AG 06, FB Mathematatik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 14: Exercises[1]

In this tutorial, we shall study some properties of eigenvalue problems with Dirichlet or Neumann boundary value. Let H be a real Hilbert space, with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$.

1. Let $\Omega_1,\ \Omega_2$ be bounded open sets of $\mathbb{R}^n$ with $n \in \mathbb{N}$ satisfying

$$\Omega_1 \subset \Omega_2.$$

Let $u \in \overset{\circ}{H}_1(\Omega_1)$. Prove that the function $\hat{u} : \Omega_2 \to \mathbb{C}$ defined by

$$\hat{u}(x) = \begin{cases} u(x), & x \in \Omega_1, \\ 0, & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

belongs to $\overset{\circ}{H}_1(\Omega_2)$.

2. Let Ω_1, Ω_2 be the same as in Problem 1. Let $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues for the Dirichlet eigenvalue problem in a domain Ω to the operator $-L_D = -\Delta$, denoted by $\lambda_k = \lambda_k(\Omega)$.

i) Prove that

$$\lambda_1(\Omega_2) \leq \lambda_1(\Omega_1).$$

ii) Prove that for any $k \in \mathbb{N} \setminus \{1\}$, there holds

$$\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1).$$

3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f \in L_2(\Omega)$. Remember that $u \in H_1(\Omega)$ is a weak solution of the Neumann problem

$$-\Delta u - \lambda u = f \text{ in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{2}$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H. D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

if

$$B(u,v) - \lambda(u,v) = (f,v), \ \forall v \in H_1(\Omega),$$

where

$$B(u,v) = (\nabla u, \nabla v).$$

i) Let $L : \mathrm{D}(L) \subset L_2(\Omega) \to L_2(\Omega)$ be the operator corresponding to Δ defined in Section 9.1 in the lecture notes. Show that $u \in H_1(\Omega)$ is a weak solution of problem (1) – (2) if and only if

$$-Lu - \lambda u = f.$$

ii) Prove Lemma 9.2 of the script for this operator L.

The following problem is your work for the semester break.

4. Prove Theorems 9.3, 9.4, Corollary 9.5, Theorem 9.6 and Corollaries 9.7, 9.8 for the operator in Problem 3.