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AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory

Tutorial 13: Exercises[1]

In this tutorial we shall investigate the definition of weak solutions to a nonlinear Dirichlet boundary value problem and the ellipticity of some partial differential operators.

1. Let $\Omega \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ and smooth boundary be a bounded open set and $f : \Omega \to \mathbb{R}$ be a function in $L_2(\Omega)$. Consider the Dirichlet boundary value problem

$$\Delta u - u^3 = f, \text{ in } \Omega, \quad (1)$$

$$u|_{\partial\Omega} = u^{(b)}, \quad (2)$$

where $u^{(b)} \in H_1(\Omega)$.

Give the definition of weak solution u to problem (1) – (2).
(**Hint.** Seek weak solution u in the space $H_1(\Omega) \cap L_6(\Omega)$.)

2. Investigate whether the following partial differential operators are elliptic, strongly elliptic, uniformly elliptic:

i)

$$L_1 u(x_1, \cdots, x_n, t) = \frac{\partial^2}{\partial t^2} u - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.$$

ii)

$$L_2 u(x_1, \cdots, x_n, t) = \frac{\partial}{\partial t} u - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u.$$

iii)

$$L_3 u(x, y) = \frac{\partial^2}{\partial x^2} u + 2i \frac{\partial}{\partial x} u \frac{\partial}{\partial y} u - \frac{\partial^2}{\partial y^2} u.$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

iv)

$$L_4 u(x,y) = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u\right) + \frac{\partial^2}{\partial y^2}u$$

in the domain a) $\Omega_1 = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$, or b) $\Omega_2 = \mathbb{R}^2$.

3. Suppose that $Lu(x) = \sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(a_{\alpha\beta}D^\beta u(x)\right)$ with $a_{\alpha\beta} = \text{const}$ and with

$$a_{\alpha\beta} = \overline{a_{\beta\alpha}}.$$

Show that the operator can be rewritten in the form

$$\tilde{L}u(x) = \sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(c_{\alpha\beta}D^\beta u(x)\right)$$

with $c_{\alpha\beta} \in \mathbb{R}$ and with

$$c_{\alpha\beta} = c_{\beta\alpha}.$$

The following problem is your homework.

4. Write the operator

$$Lu(x,y) = \frac{\partial}{\partial x}\left((x+iy)\frac{\partial}{\partial y}u(x,y)\right) + \frac{\partial}{\partial y}\left((x+iy)\frac{\partial}{\partial x}u(x,y)\right)$$

in the form

$$Lu(x,y) = \sum_{|\alpha|=1,|\beta|=1} D^\alpha\left(c_{\alpha\beta}(x,y)D^\beta u(x,y)\right)$$

with $\alpha, \beta \in \mathbb{N}_0^2$.

Check whether the symmetry conditions

$$a_{\alpha\beta}(x,y) = (-1)^{|\alpha+\beta|}\overline{a_{\beta\alpha}}(x,y).$$

is satisfied?