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AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory

Tutorial 12: Exercises[1]

This tutorial is concerned with the properties of bounded linear operators, and of compact operators.

1. Let X be a Banach space. Let $K : X \to X$ be a linear operator. K is called bounded, if there is $C > 0$ with

$$\|Kx\| \leq C\|x\|$$

for all $x \in X$. For a bounded operator define

$$\|K\| = \sup_{x \in X, x \neq 0} \frac{\|Kx\|}{\|x\|}.$$

The number $\|K\|$ is called the norm of K. Check that this is a norm on the linear space of all bounded linear operators $K : X \to X$.

2. Let X be a Banach space and $\mathcal{B}(X, X)$ be the space of bounded linear operators $K : X \to X$ with a norm defined in problem 1. Prove that $\mathcal{B}(X, X)$ is complete with this norm.

3. Assume that $K : X \to X$ is a bounded linear operator with

$$\|K\| < 1.$$

Show that $I - K$ is invertible and

$$(I - K)^{-1} = \sum_{n=0}^{\infty} K^n.$$

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

The following problem is your homework.

3. Let $K = K(x, y)$ be a real or complex-valued continuous function defined in the closure of a bounded domain $\Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$. Show that the integral operator K defined by

$$(Kf)(x) = \int_{\bar{\Omega}} K(x, y) f(y) dy$$

is compact as an operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$.

4. The product of a compact operator with a bounded operator is compact. More precisely, let $T \in \mathcal{B}(X, Y)$ be compact, and let $A \in \mathcal{B}(Y, Z)$ and $B \in \mathcal{B}(W, X)$ be bounded. Then $AT \in \mathcal{B}(X, Z)$ and $TB \in \mathcal{B}(W, Y)$ are compact.