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AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory

Tutorial 10: Exercises[1]

In this tutorial we are going to prove that the Green function to the Helmholtz equation is symmetric, the existence of fundamental solution to an operator which is a little bit more general than the Laplacian, and to show that the Green function to the Helmholtz equation with suitable boundary condition is positive.

1. Let $G = G(x, y)$ be the Green function to the Helmholtz equation with $\lambda \in \mathbb{R}$ and the Dirichlet boundary condition, prescribed on the boundary of a bounded open domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. Show that G is symmetric, namely,

$$G(x, y) = G(y, x), \quad \forall x, y.$$

(**Hint:** Define $u(z) = G(z, x)$, $v(z) = G(z, y)$ which have singularity at $z = x$ or $z = y$, respectively. Then apply the Green formula

$$\int_\Omega (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} \left(v\frac{\partial}{\partial n_y} u - u\frac{\partial}{\partial n_y} v \right) dS_y$$

(here n_y is the normal vector) to u, v over the domain $\Omega_\varepsilon := \Omega \backslash \{B_\varepsilon(x) \cup B_\varepsilon(y)\}$, and let $\varepsilon \to 0$.)

2. Let $\Omega \subset \mathbb{R}^3$ be a bounded open domain with smooth boundary, and let $G = G(x, y)$ be the Green function to the Helmholtz equation

$$\Delta u + \lambda u = 0, \text{ in } \Omega$$

with the Dirichlet boundary condition, where $\lambda < 0$.

Prove that G is non-negative, i.e.

$$G \geq 0.$$

[1] If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

The following problem is your homework.

Now let us give a definition:
Definition Let

$$A=\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(a_i\frac{\partial}{\partial x_i}\right)$$

with $a_i,\ i=1,2,3$ being positive constants. We call a function F a *fundamental solution* to an equation $Au=0$ in $\mathbb{R}^n$ where A is a differential operator, if F is infinitely differentiable for $x\neq y$,

$$A_xF(x,y)=0,\quad A_yF(x,y)=0,$$

and

$$\lim_{r\to 0}\int_{\{y\in\mathbb{R}^3|\ |x-y|\ =\ r\}}\frac{\partial}{\partial n_y}F(x,y)\,dS_y=1,$$

where n_y is the unit exterior normal vector.

3. Prove that the fundamental solution $F=F(x,y)$ to $Au=0$ in $\mathbb{R}^3$ is

$$F=\frac{1}{\kappa}\cdot\frac{1}{|x-y|_M},$$

here $|x-y|_M:=\sqrt{(x-y)\cdot M(x-y)}$ and M is a matrix defined by

$$M=\begin{pmatrix} a_1^{-1} & 0 & 0\\ 0 & a_2^{-1} & 0\\ 0 & 0 & a_3^{-1}\end{pmatrix},$$

and κ is a constant which is equal to $\int_{|\omega|=1}\frac{1}{|\omega|_M}d\omega$. In the case that the operator is the Laplacian, i.e. $M=Id$, then $\kappa=4\pi$ as in the lecture notes.
Remark. We can prove similar result for a more general operator

$$A=\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right),$$

where a_{ij} are constant and such that $a_{ij}=a_{ij}$ for all $i,j=1,2,3$.

Merry Christmas and Happy new year!