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AG 06, FB Mathematik
Tech. Univ. Darmstadt

Partial Differential Equations I: Linear Theory
Tutorial 09: Exercises[1]

This tutorial is concerned with the existence of solutions to the Helmholtz equation in some special domains, the properties of the solution with special boundary condition, and with the application of the fundamental functions to the Laplace equation.

1. Let $\Omega_\alpha = \{(r,\varphi) \mid r > 0, \quad 0 < \varphi < 2\pi - \alpha\}$ with $0 \le \alpha < 2\pi$. Draw a picture of Ω_α.

i) Find a solution $u \neq 0$ of

$$\begin{aligned} \Delta u + u &= 0, \text{ in } \Omega_\alpha, \\ u|_{\partial\Omega_\alpha} &= 0. \end{aligned}$$

ii) Show that there is a real valued solution such that

$$u(r,\varphi) = r^{\frac{\pi}{2\pi-\alpha}} \sin\left(\frac{\pi}{2\pi-\alpha}\varphi\right)(1 + O(r)).$$

Here O is the so-called big "O", the Landau symbol.

This shows that u has an interesting asymptotic behavior at the tip of the cone, depending on the opening angle α.

2. Assume that $f \in L_2(\mathbb{R}^3)$ with $f(x) = 0$ for all $|x| \ge 1$, and $u = F * f$ where F (assume it is real and $\lambda = 0$, for simplicity) is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$.

Prove that

$$\lim_{|x|\to\infty} |x|u(x) = \frac{1}{4\pi}\int_{|x|\le 1} f(x)dx.$$

[1] If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

The following problem is your homework.

3. (**Removable singularity**) Let $B_1(0)$ be a ball in $\mathbb{R}^n (n \geq 2)$. Assume that a real function $u \in C^2_{\text{loc}}(B_1(0)\backslash\{0\})$ is bounded and harmonic in $B_1(0)\backslash\{0\}$. Prove that if the limit $\lim_{x\to 0} u(x) =: a$ exists, and if we define $u(0) = a$, then u is harmonic in $B_1(0)$.

(**Hint:** Let v be the solution to $\Delta v = 0$ with $v|_{\partial B_1(0)} = u|_{\partial B_1(0)}$. Then define $w_{\pm} = u - v \pm \varepsilon G(r)$ which tends to $\pm\infty$ as $x \to 0$, where $G(r)$ is the nonnegative fundamental solution to the Laplace equation and ε is any positive constant. Applying the maximum principle we find $w_{\pm} \geq 0$ for all $x \in B_1(0)\backslash\{0\})$ for arbitrary ε. Then $u \equiv v$ follows.)