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Partial Differential Equations I: Linear Theory
Tutorial 08: Exercises[1]

In this tutorial we are going to discuss the converse of mean value formula, some properties of the (sub-) harmonic functions. We also give exercisee on construction a sub-solution or a super-solution to the Dirichlet problem for the Helmholtz equation or the Laplace equation.

1. (**Converse of the mean value formula**) Let $\Omega \subset \mathbb{R}^n$. Show that function $u \in C^2(\Omega)$ is harmonic if u satisfies the mean value formula.

(**Hint:** Assume that there exists a point $x_0 \in \Omega$ such that $\Delta u(x_0) \neq 0$. Then use the Gauss formula.)

2. Suppose that u is a harmonic function. Prove that
i) u^2 is subharmonic.
ii) More generally, for any convex function $f \in C_2(\mathbb{R}, \mathbb{R})$, the composite function $f(u)$ is subharmonic.

3. In this problem we assume $\Omega \subset \mathbb{R}^2$. Prove that $u(x) = |x|$ is a subsolution to the equation $\Delta u = 0$.

(**Hint:** Show that the two-times continuously differentiable function $x \mapsto \sqrt{|x|^2 + \varepsilon^2}$ with $\varepsilon > 0$, is a subsolution and consider the limit as $\varepsilon \to 0$.)

The following problem is your homework.

4. i) Construct a supersolution w to the equation

$$\lambda v + \Delta v = 0, \quad \text{in} \quad \Omega,$$

with $\lambda \leq 0$ satisfying the boundary condition $w|_{\partial\Omega} = 0$.

[1]If you have any opinion and/or suggestion on the Tutorial, please send your email to Prof. Dr. H.-D. Alber at alber@mathematik.tu-darmstadt.de, or to Dr. P. Zhu at zhu@mathematik.tu-darmstadt.de.

ii) Assume that the boundary $\partial\Omega$ is C^2-smooth. Construct a subsolution v to this equation satisfying $v|_{\partial\Omega} = 0$.

(**Hint:** For ii) we consider the function in the form:

$$u_1(x) = -M\left(1 - e^{-\mu d(x)}\right)$$

with M, μ being positive constants, and $d(x) = \inf_{y\in\partial\Omega} |x - y|$, the distance of the point x and the boundary. Then we define $\underline{u}(x) = \max\{u_1, -C\}$ near $\partial\Omega$; $\underline{u}(x) = -C$ Otherwise in Ω, where C is a suitable positive constant.

Since the boundary $\partial\Omega$ is C^2, we have that $d(x)$ is C^2 too for x near $\partial\Omega$.)